

# Elastic waves in one dimension

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## WAVES IN A TORSIONAL OSCILLATOR

In class we have investigated the physics of torsional wave machine: a set of rods pivoted on a straight wire (Fig. 1). This is a great model to study elastic waves in one spatial dimension. In this assignment we will examine standing waves and associated with them resonances of the torsional oscillator with the aid of a numerical simulation written in Java.

Let us quickly review the basics of the torsional oscillator. If the rods were pivoted through their centers of mass, the torque acting on the  $n$ -th rod would be determined by the twist of the wire between that rod and rods  $n - 1$  and  $n + 1$ . To the lowest order, such torques would be proportional to the angle difference  $\theta_{n\pm 1} - \theta_n$ , hence the equations of motion

$$I\ddot{\theta}_n = C(\theta_{n+1} - \theta_n) + C(\theta_{n-1} - \theta_n),$$

where  $I$  is the rod's moment of inertia and  $C$  is a constant characterizing the twisting modulus of the wire. For convenience we combine the two unknown physical constants  $I$  and  $C$  into one,  $s^2 = C/I$ :

$$\ddot{\theta}_n = s^2(\theta_{n+1} - 2\theta_n + \theta_{n-1}). \quad (1)$$

As will become clear later,  $s$  determines the speed of waves in the torsional oscillator.

According to Eq. (1), the oscillator will remain stationary if all rods are rotated uniformly:  $\theta_1 = \theta_2 = \dots = \theta_N$ . The wire is not twisted, so there should be no torque. The actual torsional oscillator does not behave that way: the wire is attached to the rods at the top, so that when the rods are rotated away from the horizontal position their centers of mass are raised and gravity creates a restoring torque. To fix this discrepancy of the model (1), we add the term  $-\omega_0^2\theta_n$  on the right-hand side:

$$\ddot{\theta}_n = s^2(\theta_{n+1} - 2\theta_n + \theta_{n-1}) - \omega_0^2\theta_n. \quad (2)$$

If we now rotate the rods uniformly, they will oscillate with the frequency  $\omega_0$  about their horizontal positions, as we observed in class.



FIG. 1: Torsional wave machine.

### Continuum approximation

Eq. (2) is a set of coupled linear differential equations. Although such systems can be solved in full detail, the mathematics simplifies considerably if we regard the discrete set of rods as a continuum. This is justified when the angles  $\theta_n$  vary slowly with  $n$ . Then we can treat the discrete index  $n$  as a continuum variable  $x$ . (Such approximations are often made when we deal with crystals: though they are made of discrete atoms, in practice deformations vary slowly from one atom to the next.) We can then expand  $\theta(x \pm 1)$  in a Taylor series and keep the lowest-order terms:

$$\theta(x \pm 1) = \theta(x) \pm \theta'(x) + \theta''(x)/2 + \dots$$

Equation (2) becomes a partial differential equation:

$$\ddot{\theta} = s^2\theta'' - \omega_0^2\theta. \quad (3)$$

Here the dot signifies the time derivative  $\partial/\partial t$ , while the prime means the spatial derivative  $\partial/\partial x$ .

Equation (3) has solutions in the form of right and left-running waves,

$$\theta(x, t) = A_R \cos(\omega t - kx), \quad \theta(x, t) = A_L \cos(\omega t + kx).$$

Substituting either of these into Eq. (3) yields a relation between the frequency of the wave  $\omega$  and wavenumber  $k$ , often called the *dispersion relation*:

$$\omega^2 = \omega_0^2 + k^2. \quad (4)$$

In particular, we recover a familiar result: for uniform oscillations ( $k = 0$ ) the frequency  $\omega = \omega_0$ .

### Forbidden band

The dispersion (4) has an interesting feature: for frequencies  $\omega < \omega_0$  it predicts an *imaginary* wavenumber  $k = is^{-1}\sqrt{\omega^2 - \omega_0^2}$ .

**Problem 1.** Show that the wave equation (3) admits solutions exponentially decaying in space:  $\theta(x, t) = A \cos \omega t e^{-\kappa x}$ , where  $\kappa = s^{-1}\sqrt{\omega_0^2 - \omega^2}$  is real and positive.

The exponential decay does not mean strong absorption of these waves by the medium. Rather, it is an indication that waves with frequencies below  $\omega_0$  are unable to propagate through our wave machine and are “reflected back.” The frequency band below  $\omega_0$  is thus forbidden.

Forbidden bands exist in other physical systems. For example, an electromagnetic wave cannot propagate through a plasma if its frequency is below a characteristic value, the *plasma frequency*. For this reason metals (containing electron plasma) reflect visible light but are transparent in the UV. Similarly the ionosphere—an ionized layer of the Earth’s atmosphere—reflects slow radiowaves but is transparent to fast ones. As a result, low-frequency (less than 30 MHz) radio signals can literally travel around the globe by bouncing off the ionosphere, while waves of higher frequency can only be received within the direct line of sight.

Photonic crystals also possess allowed and forbidden frequency bands for light. And, last but not least, real atomic crystals have allowed and forbidden energy bands for electrons.

A wave with a forbidden frequency penetrates only a finite distance into the medium. In total internal reflection of light this length can be a fraction of a micron. In our wave machine, where a forbidden wave decays as  $e^{-\kappa x}$ , the penetration length is  $1/\kappa = s/\sqrt{\omega_0^2 - \omega^2}$ . As we approach the edge of the forbidden band,  $\omega \rightarrow \omega_0$ , the penetration length increases and becomes infinite for  $\omega = \omega_0$ . Waves with higher frequencies can freely propagate through the medium.

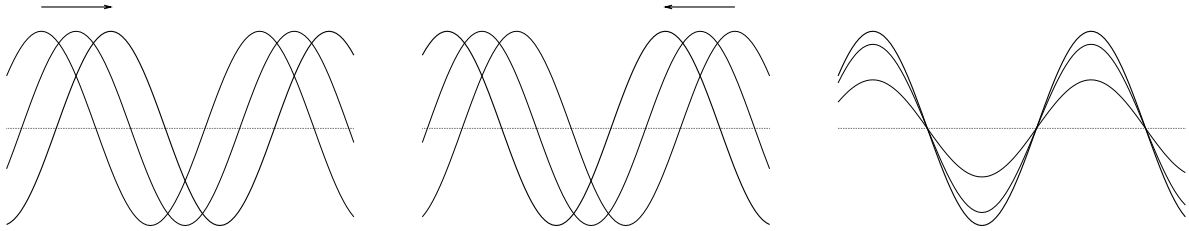


FIG. 2: Right-moving, left-moving, and standing waves.

### Boundary conditions and standing waves

Our wave machine has edges. As we discussed in class, waves do not disappear at an edge: they are reflected back with an undiminished amplitude. If waves propagate freely through the machine, they reflect multiple times from its edges. We have to take into account these multiple reflections. This sounds like a difficult problem: reflections accumulate as a wave propagates. However, there is a way to take into account multiple reflections in one fell swoop. Instead of focusing on individual reflections we will think about their underlying cause: boundary conditions at the edges.

We can anticipate the result by noting that the reflection of a left-traveling wave  $A \cos(\omega t + kx)$  from the edge at  $x = 0$  is a right-traveling wave of the same amplitude  $\pm A \cos(\omega t - kx)$ . (We'll see below what determines the sign.) Adding the two waves yields, depending on the sign of the reflected wave,

$$\theta(x, t) = 2A \cos \omega t \cos kx \quad \text{or} \quad \theta(x, t) = -2A \cos \omega t \sin kx.$$

These solutions are called standing waves because the crests and troughs are not moving. Standing waves satisfy exactly the same dispersion relation (4) as do traveling waves.

In finite systems, where boundary conditions play a major role, thinking in terms of standing waves is more convenient. For example, if the first rod is clamped down,  $\theta(0, t) = 0$ , the sine standing waves  $A_s \cos \omega t \sin kx$  automatically satisfy this boundary condition. If, on the other hand, the first rod is free, there is no torque acting on this rod from the left. We can formulate the boundary condition in this case by imagining that there is a fictitious rod number 0. It won't apply any torque to rod 1 provided that  $\theta_1 - \theta_0 = 0$ . In our continuum approximation that translates into  $\theta'(x) = 0$  at  $x = 0$ . It is easy to see that the cosine standing waves  $A_c \cos \omega t \cos kx$  satisfy this condition.

Our system, in fact, has two ends, so we must impose two boundary conditions:  $\theta'(0) = 0$ ,  $\theta'(L) = 0$ . Here  $L$  is the length of the system. A cosine standing wave automatically satisfies the first of these. The second boundary condition yields a restriction on possible wavenumbers  $k$ :

$$\sin kL = 0.$$

The allowed wavenumbers are [1]

$$k_n = \pi n/L, \quad n = 0, 1, 2, \dots$$

The first three of these modes are shown in Fig. 3.

These frequencies will appear as *resonances* in the response of the system to a periodic external perturbation. In other words, the amplitude of oscillations will be strongly enhanced when you rock the wave machine at one of its natural frequencies.

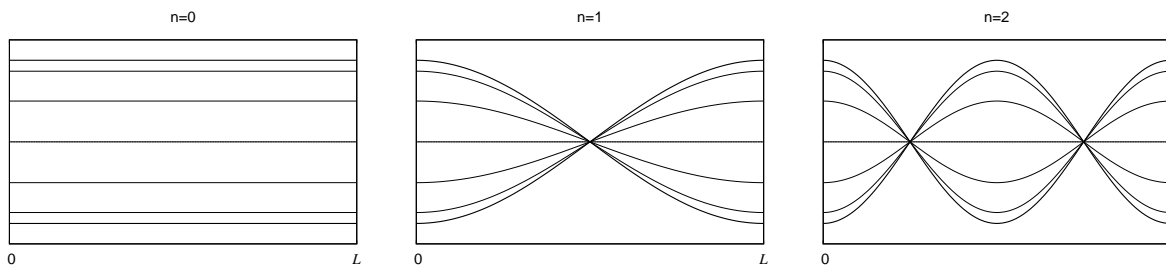


FIG. 3: Standing waves  $\theta(x,t) = A \cos \omega t \cos(\pi n x/L)$  in a system with open boundary conditions with  $n = 0, 1,$  and  $2$ .

## THE SIMULATION

The Java applet simulates a torsional wave machine with  $L = 12$  rods. Rod moments of inertia are  $I = 1.0$ , the elastic constant  $C = 1.0$ , and the frequency of uniform oscillations  $\omega_0 = 0.5 \text{ s}^{-1}$ .

The main screen gives you a three-dimensional view of the wave machine or a two-dimensional projection showing rod endpoints (use the radio button to switch). The first rod on the left (shown in red) is driven by an external torque  $\tau = \tau_0 \cos \omega t$ . The external frequency  $\omega$ , shown just below the main window, is slowly increasing in time:  $\omega(t) = \omega(0) + \gamma t$ . Both the frequency  $\omega$  and the sweep rate  $\gamma$  can be adjusted by typing in the desired values and pressing ENTER. (It might be a good idea to pause the simulation as you adjust the frequency.) Clicking on the rightmost rod in the 2D view toggles between the free and clamped states of the right edge.

A second window shows the energy of the wave machine vs the current frequency of external oscillations. This allows you to observe the resonances of the system: the energy will exhibit sharp maxima at the resonant frequencies. Clicking on the screen gives you the coordinates of the pointer thus enabling you to determine the resonant frequencies with a good precision.

The default sweep rate should be okay for a quick first look at the simulation. Be warned, however, that a high sweep rate may broaden the resonance peaks [2]. The broadening effect will be particularly detrimental at the beginning of the spectrum where the resonant frequencies  $\omega_0$  and  $\omega_1$  are quite close to each other. If your sweep rate is too fast you won't be able to resolve the two peaks.

Since it takes time to build a resonant response, you will encounter a lag between the external frequency and the response of the system. It is therefore helpful to sweep in both directions. The resonance peaks will be shifted in opposite directions with the "true" position in the middle.

## SUGGESTED ACTIVITIES

**Problem 2.** Using the applet, estimate the penetration length of the waves into the machine. Compare with your earlier theoretical estimate.

**Problem 3.** Compute the first few frequencies for free ends. Do the same for one free end and the other fixed. Use the parameters given above.

**Problem 4.** Using the applet, measure the resonant frequencies for both ends free, as well as for one end free and one fixed end.

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- [1] *Caution:* In the continuum approximation the wave index  $n$  increases without limit giving shorter and shorter wavelengths. In our machine, consisting of  $L$  discrete rods, we can have at most  $L$  different modes. Therefore the continuum approximation is expected to fail when  $n$  approaches  $L$  and the wavelength  $\lambda_n = 2\pi/k_n = 2L/n$  becomes of order 1 (the distance between adjacent rods). Still, it should provide an adequate description for the first few long-wavelength modes, for which the discrete nature of this “medium” is not important.
- [2] If you pass through a resonance too quickly, the system may not have enough time to develop an enhanced response. Additionally, passing too quickly from one resonance frequency to another means that the previous resonant mode has not had enough time to decay and the response to one resonant frequency is contaminated by a previous resonance.