Field Quantization

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I. CANONICAL QUANTIZATION

Well. let us start from the simplest case – free fields.

Canonical quantization is not something new, but rather borrowed from non-relativistic quantum mechanics. To see this clear, one may notice this quantization procedure lost Lorentz invariance.
So, here is the question, when should we use commutation or anti-commutation relation.

A. EXAMPLE I: Bosonic string (non-relativistic case)

The Classical Lagrangian is

$$H = \frac{1}{2}m\omega^2 X^2 + \frac{1}{2}\frac{P^2}{m} = \omega\left[\left(\sqrt{\frac{m\omega}{2}}X\right)^2 + \left(\sqrt{\frac{1}{2m\omega}}P\right)^2\right]$$

We know the trick to solve this problem, by defining

$$a = \sqrt{\frac{m\omega}{2}}X + i\sqrt{\frac{1}{2m\omega}}P$$

$$a^* = \sqrt{\frac{m\omega}{2}}X - i\sqrt{\frac{1}{2m\omega}}P$$

The Lagrangian will become:

$$H = \omega a^* a = \omega(aa^* + a^* a)/2$$

Now, we consider to quantize this system:

$$H = \frac{1}{2}m\omega^2 X^2 + \frac{1}{2}\frac{\hat{P}^2}{m} = \omega\left[\left(\sqrt{\frac{m\omega}{2}}X\right)^2 + \left(\sqrt{\frac{1}{2m\omega}}\hat{P}\right)^2\right]$$

By analogy, we can define(in X-repr)

$$\hat{a} = \sqrt{\frac{m\omega}{2}}X + i\sqrt{\frac{1}{2m\omega}}P = \sqrt{\frac{m\omega}{2}}X + \sqrt{\frac{1}{2m\omega}}\frac{\partial}{\partial X}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}X - i\sqrt{\frac{1}{2m\omega}}\hat{P} = \sqrt{\frac{m\omega}{2}}X - \sqrt{\frac{1}{2m\omega}}\frac{\partial}{\partial X}$$

Now the Lagrangian becomes:

$$H = \omega(aa^\dagger + \frac{1}{2})$$

Obviously, this quantum system describe all excitation on the background.

B. EXAMPLE II: Non-relativistic Fermionic String

A drawback of "SHO" analogy is lacking of Fermionic part. First, let us just do it.
The Fermionic system in 1+1 dim has Lagrangian:

$$\mathcal{L} = \bar{\lambda}\partial\!\!\!/\lambda - m\lambda\lambda$$

We only need two gamma matrices, they can be chosen as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then the spinor $\lambda = (\xi, \eta)^T$ will satisfy Dirac equation:

$$i(\partial_0 - \partial_1)\eta = m\xi$$

$$i(\partial_0 + \partial_1)\xi = m\eta$$

Each component satisfy K-G equation or wave equation

$$(\partial_0^2 - \partial_1^2 + m^2)\chi = 0$$

As a special case, massless equation has solution

$$\xi = f(x+t); \qquad \eta = g(x-t)$$

Thus massless 2D spnior split into "left moving" component and "right moving" one.

Do we lost generality here?
No, because we can only measure "probability" of spinor states. the handness are totally artificial, in 2D, when we choose handness, the direction of motion is also determined.
Go back to general massive case, by comparing coefficient, we can expand general solution as

$$\xi(x,t) = \sum_k (a_k e^{ikx} + b_k e^{-ikx}) e^{-iEt}$$

$$\eta(x,t) = \sum_k \left(\frac{(E-k)}{m} a_k e^{-ikx} + \frac{(E+k)}{m} b_k e^{ikx} \right) e^{-iEt}$$

let us check the second special case: chargeless fermion (neutron). It turns out in massive case real condition can not be satisfied by both ξ and η, something is wrong here.
So, what is wrong?
Answer: **frequency**!

In fermionic case, we can not just define "positive-energy" sets. negative frequency solution must be included to make a complete set!
So, we should choose solution as

$$\xi(x,t) = \sum_k (a_k e^{ikx-iEt} + b_k e^{-ikx+iEt})$$

$$\eta(x,t) = \sum_k \left(\frac{(E-k)}{m} a_k e^{ikx-iEt} - \frac{(E-k)}{m} b_k e^{-ikx+iEt} \right)$$

A naive checking show this time right hand component η is always imaginary. Is this a problem? Obviously not. Only the relative phase does matter. The imaginary condition is fine and consists with Dirac equation.

Now, by these two simplest case, I have shown you how to quantize spinless and spin one-half field. The only difference is in spinless case, we can think negative energy state as backwards moving anti-particle and just flip the time direction, while in fermionic case, we must be careful and really include negative frequency states. The reason can be simply stated as fermion is not twice time flip invariant(you need exact minus 1 phase).

C. EXAMPLE III: Commutator and Anti-commutator

Canonical quantization start with Possion bracket in classical mechanics.

$$\{f,g\}_{PB} = \left(\sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}\right)$$

Here, we first assume f and g are "functions", it turns out later they must further be bosonic operators. Canonical quantization rule for (bosonic) quantum mechanics is:

$$\{f,g\}_{PB} \longrightarrow \frac{[f,g]}{i\hbar}$$

However, things are different when it comes to fermionic case. Let us try this(still in 2D):

$$A = \gamma^0 X; \qquad B = \gamma^1 \partial$$

$$\{A,B\}_{PB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So Possion bracket of A, B is well defined in classical mechanics.
However, if we try quantum commutator:

$$[A,B] = \left[\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix} - \begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\right] = \begin{pmatrix} -X\partial - \partial X & 0 \\ 0 & X\partial + \partial X \end{pmatrix}$$

which leads us to nowhere.
Instead, anti-commutator obtains the correct relation:

$$\{A,B\} = \left[\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial \\ -\partial & 0 \end{pmatrix}\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}\right] = i\hbar\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus we believe in Fermionic case, the canonical quantization rule is:

$$\{A,B\}_{PB} \longrightarrow \frac{\{A,B\}}{i\hbar}$$

According to this example, we know this different is not new in field theory. Once you chose Fermionic operator(which is expanded in fermionic basis like gamma matrices), you must use anti-relation. It is a "base-depend" principle.

D. TEXTBOOK CHOICE

The most importance is physics instinct. Other than that are just labor. Of course you also need tricks and mathematical manner.

Peskin states everything he wrote down clear, Weinberg try to show everything he knew clear. Whatever you choose to read, "***Physics is Physics***".