Quantum Mechanics (171.605),
Fall 2016

Problem Set 9

Due: 13 December 2016

1. Problem 4.11 in Sakurai-Napolitano.


3. Determine the energy levels for a particle of mass \( m \) moving in one-dimensional lattice of Dirac delta functions; i.e.,

\[
V(x) = \sum_{n=-\infty}^{\infty} V_0 \delta(x - na). \tag{1}
\]

4. The Hamiltonian of a positronium atom in the 1\( S \) state in a magnetic field \( B \) along the \( \hat{z} \) axis is to good approximation,

\[
H = A \vec{S}_1 \cdot \vec{S}_2 + \frac{eB}{mc}(S_{1z} - S_{2z}), \tag{2}
\]

if all higher energy states are neglected. The electron is labeled as particle 1 and the positron as particle 2. Using the *coupled representation* in which \( \vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2 \) and \( S_z = S_{1z} + S_{2z} \) are diagonal, obtain the energy eigenvalues and eigenvectors and classify them according to the quantum numbers associated with the constants of the motion. Empirically it is known that for \( B = 0 \) the frequency of the \( 1^3S \to 1^1S \) transition is \( 2.0338 \times 10^5 \) MHz and that the mean lifetimes for annihilation are \( 10^{-10} \) sec for the singlet state (two-photon decay) and \( 10^{-7} \) for the triplet state (three-photon decay). Estimate the magnetic-field strength \( B \) which will cause the lifetime of the longer lived \( m = 0 \) state to be reduced (or “quenched”) to \( 10^{-8} \) sec.

5. The spin-angular function for a spin-1/2 particle in a state of total angular momentum \( j = l \pm 1/2 \) and azimuthal quantum number \( m \) is

\[
\mathcal{Y}^{l \pm 1/2, m} = \frac{1}{\sqrt{2l+1}} \left( \begin{array}{c} \pm \sqrt{l \pm m + \frac{1}{2}} Y_{l,m-(1/2)} \\ \sqrt{l \mp m + \frac{1}{2}} Y_{l,m+(1/2)} \end{array} \right). \tag{3}
\]

Now suppose that this spin-1/2 particle is in a state,

\[
\mathcal{Y}^{jm} j_{-(1/2)} + b \mathcal{Y}^{jm} j_{+(1/2)}, \tag{4}
\]

with total angular momentum \( jm \). Assume this state to be an eigenstate of the Hamiltonian \( H \) with no degeneracy other than that demanded by
rotational invariance. If $H$ conserves parity, how are the coefficients $a$ and $b$ restricted? If $H$ is invariant under time reversal, show that $a/b$ must be imaginary. Verify explicitly that the expectation value of the dipole operator $-e\mathbf{r}$ vanishes if either parity is conserved or if time-reversal invariance holds (or both).