Week 1: Classical Cosmology

January 17, 2017

1 The expansion and Friedmann-Robertson-Walker metric

Hubble discovered in the late 1920s that every galaxy moves away from us at a recessional velocity $v \equiv cz$ (where c is the speed of light and z is the "redshift") proportional to its distance $d_{\rm L}$: i.e., $v = H_0 d_{\rm L}$, where $H_0 = 100\,h$ km/sec/Mpc is the Hubble parameter, and current measurements give us $h \simeq 0.7$ to a few percent. It is also observed that on the largest observable scales (100s to 1000s of Mpc) the galaxy distribution is roughly isotropic; we see roughly the same distribution of galaxies in every direction we look—later this quarter we will quantify this more precisely. If we then apply the Copernican principle, which states that we do not occupy a preferred position in the Universe, then it follows that any observer in any other galaxy should also see an isotropic distribution of galaxies. This can only be accommodated if the Universe is, in addition to being isotropic, homogeneous.

The Hubble law $v = H_0 d$ and the assumption of homogeneity can only be reconciled if the relative velocity between any two galaxies in the Universe is proportional to the distance between them. This situation can be realized mathematically in a spacetime with the Friedmann-Robertson-Walker metric,

$$ds^2 = dt^2 - [a(t)]^2(dx^2 + dy^2 + dz^2), \tag{1}$$

where $a(t)$ is the scale factor, a function of time t. This metric describes a Universe with a three-dimensional Euclidean subspace, with coordinates x, y, and z, that expands (or contracts) in a manner described by the scale factor. It yields the Hubble law $v = Hr$ for the relation between the relative velocity v and distance r between any two objects as long as the Hubble parameter is $H(t) = \dot{a}/a$, where the dot denotes a derivative with respect to time t.

In the limit that $|v| \ll c$, the redshift z describes the Doppler shift between the wavelength λ_e of light emitted by galaxy "e" and the wavelength λ_o observed by galaxy "o": $z \equiv v/c = H_0\vec{r}/c = (\lambda_{\rm o}/\lambda_{\rm e}) - 1$.

Eq. (1 describes a "flat" FRW Universe, just one of three possible spacetimes that satisfy the requirements of homogeneity and isotropy. The other two are obtained by generalizing to

$$ds^2 = dt^2 - [a(t)]^2 \left(\frac{dr^2}{1 - kr^2} + r^2\, d\theta^2 + r^2 \sin^2\, d\phi^2 \right), \tag{2}$$

in (r, θ, ϕ) spatial coordinates, or

$$ds^2 = dt^2 - [a(t)]^2 \left[d\chi^2 + \begin{pmatrix} \sinh^2 \chi \\ \chi^2 \\ \sin^2 \chi \end{pmatrix} \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \tag{3}$$

in (χ, θ, ϕ) coordinates for (from top to bottom) $k = -1$, $k = 0$, and $k = 1$. The flat Universe is recovered with $k = 0$. The cases $k = 1$ and $k = -1$ describe, respectively, a closed and open Universe in which the three-dimensional subspaces are surfaces of constant positive and negative curvature, respectively. The spatial slices of a closed Universe, in particular, are three-spheres, S^3, and thus have finite volume. Measurements now indicate that our Universe is spatially flat, or extremely close, and so we will assume most of the time that $k = 0$. Still, it is important to consider the open and closed Universes as well to understand why it is that measurements indicate $k = 0$.

It is also often useful to define a new time coordinate, the *conformal time* η by $d\eta = dt/a(t)$. Then,

$$ds^2 = a^2(\eta) \left[d\eta^2 - \frac{dr^2}{1 - kr^2} - r^2 \, d\theta^2 - r^2 \sin^2 \theta \, d\phi^2 \right]. \tag{4}$$

This is kind of nifty, as with this time coordinate, the metric (for the flat Universe) is said to be *conformal* to the Minkowski metric; i.e., it is the same thing times some scale factor. With this metric, photons travel along the same coordinate trajectories as they would in Minkowski space. Keep in mind, though, that the conformal time is *not* the proper time—it is not the time measured by a comoving observer. That is still the original time coordinate t.

A note on normalizations: Textbooks are often careless about units and normalization in the FRW metric. My preference is to define r and χ to be dimensionless, in which case $a(t)$ has units of length (or time if you choose to leave out the c from the $\mathrm{d}t^2$ term). Then, we can set the present day value of $a(t)$ to obtain the correct radius of curvature, which means we need

$$a(t_0) = \mathrm{c} H_0^{-1} \begin{cases} \sqrt{-1/\Omega_\mathrm{k}} & \text{if } k = +1 \\ 1 & \text{if } k = 0 \\ \sqrt{1/\Omega_\mathrm{k}} & \text{if } k = -1, \end{cases} \tag{5}$$

and $\Omega_\mathrm{k} = 1 - \Omega_\mathrm{m} - \Omega_\Lambda$ is the energy density in curvature (as we'll see later).

2 Equation of motion for the scale factor

In general relativity, the matter content determines the spacetime. In a homogeneous and isotropic universe, the matter content is described by a matter density $\rho(t)$ and pressure $p(t)$, and Einstein's equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, evaluated for the FRW metric yield the first form of the Friedmann equation,

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \rho}{3} - \frac{k}{a^2}. \tag{6}$$

and its second form,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p). \tag{7}$$

The second form of the Friedmann equation can also be derived from the first form differentiating $\dot{a}$ with respect to time, and by observing that the change $d(\rho a^3)$ in the total energy per comoving volume is equal to the work $-pd(a^3)$ done by the fluid. Both forms of the Friedmann equation can also be derived using Newtonian arguments and replacing the source ρ in the Poisson equation $\nabla^2\phi = 4\pi G\rho$ for the Newtonian gravitational potential ϕ by a relativistic gravitating source density $\rho + 3p$.

We now evaluate the Friedmann equation today—i.e., evaluating the Hubble parameter at its current value H_0 and the matter density at its current value ρ_0—and divide both sides of the Friedmann equation by H_0^2. Then, define the density parameter $\Omega \equiv \rho_0/\rho_c$, where $\rho_c \equiv 3H_0^2/8\pi G$. Then, we have

$$\Omega_m = 1 + \frac{k}{a_0^2 H_0^2}. \tag{8}$$

We therefore see that the ratio of the matter density to the expansion rate (which can in principle be measured) is related to the geometry of the Universe. For $\Omega > 1$, $\Omega = 1$, $\Omega < 1$, the Universe is closed, open, or flat, respectively.

Our Universe consists of several types of matter, and so we write the matter density as $\rho = \sum_i \rho_i$, over several different components labeled by i, each of which as a pressure assumed to be $p_i = w_i\rho_i$, where w_i are equation-of-state parameters.

Nonrelativistic matter (or "dust" in older books)—things like "baryons" (cosmologists' term for stars and gas) or cold dark matter—is effectively pressureless, and so has $w = 0$. Relativistic matter, like the cosmic microwave background, (effectively) massless neutrinos, or ultra-relativistic partiles in the early Universe, has $w = 1/3$. A cosmological constant behaves as matter with $w = -1$. The first law of thermodynamics, $dE = p\,dV$ applied to each component becomes, $d(\rho_i a^3) = -p_i d(a^3)$ from which it follows that $\rho_i \propto a^{-3(1+w_i)}$. E.g., the energy density of nonrelativistic matter matter scales as $\rho_m \propto a^{-3}$, relativistic matter as $\rho_r \propto a^{-4}$, and cosmological constant as $\rho_\Lambda \propto$ constant.

Throughout much of the history of the Universe, the energy density is dominated largely by one component, and so we refer, e.g., to a matter-dominated (MD), radiation-dominated (RD), or cosmological-constant–dominated phase.

The second form of the Friedmann equation gives us also the *deceleration parameter*, $q \equiv -a\ddot{a}/\dot{a}^2$. For a Universe with nonrelativistic matter and a cosmological constant, $q_0 = \Omega_m/2 - \Omega_\Lambda$, where the subscript 0 denotes the value today, and Ω_m and Ω_Λ are here their values today. Current measurements suggest $q_0 \simeq -0.55$.

We now quickly review some simple solutions for the Friedmann equation. For a MD Universe, $a \propto t^{2/3}$; for RD, $a \propto t^{1/2}$; and for vacuum-dominated, $a \propto e^{Ht}$. The vacuum-dominated solution is known as a de Sitter spacetime. Such a spacetime has a higher degree of symmetry, there being no preferred time direction as in the other (e.g., MD or RD) FRW Universes. For an empty Universe with no cosmological constant but negative curvature, the Friedmann equation becomes $H^2 = \frac{1}{a^2}$, which has solution $a \propto t$. Such a spacetime, the *Milne spacetime*, has no matter and must therefore be equivalent to a Minkowski. This can be shown with an appropriate change of coordinates.

We define density parameters $\Omega_i = \rho_i/\rho_c$ in terms of a "critical density" $\rho_c \equiv 3H^2/(8\pi G)$. These most often refer to their values today but in fact are functions of time. The total density $\Omega \equiv \sum_i \Omega_i$ is related to the geometry of the Universe with $\Omega > 1$, $\Omega = 1$, and $\Omega < 1$ corresponding, respectively, to closed, flat, and open Universes.

3 Expansion History and $\Omega_m - \Omega_\Lambda$ plots.

Expansion times and distance scales. If we divide the galaxy separation by the relative velocity, we obtain a characteristic expansion time $t \sim r/v = H_0^{-1} \simeq 1.5 \times 10^{10}$ yr which, we will see, is quite close to the age of the Universe. Quite remarkably, this is quite close to the ages of the oldest stars, even though the timescale for stellar evolution is determined by an amalgamation of gravity and nuclear and atomic physics, and should thus should *a priori* have nothing to do with the expansion rate H_0 of the Universe. The distance that light can travel in this time is $l \sim cH_0^{-1} \simeq 5000\,\mathrm{Mpc} \simeq 1.5 \times 10^{28}\,\mathrm{cm}$, which we will later see is numerically quite close to the size of the observable Universe. Also, in order of magnitude, the volume of the Universe should be roughly $V = (4/3)\pi l^3 \sim 3 \times 10^{11}\,\mathrm{Mpc}^3$, and should thus contain roughly several hundred billion galaxies.

More Precise Age of the Universe. To determine the age of the Universe, we simply integrate the time since $t = 0$ (when $a \to 0$; the big bang) until today:

$$t_0 = \int_0^{t_0} dt = \int_0^{a_0} \frac{da}{\dot{a}}. \tag{9}$$

We then recast the Friedmann equation (assuming nonrelativistic matter and a cosmological constant) as an equation for the expansion rate as a function of redshift,

$$H(z) = \frac{\dot{a}}{a} = H_0 E(z), \tag{10}$$

where

$$E(z) = \left[\Omega_m(1+z)^3 + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)(1+z)^2\right]^{1/2}. \tag{11}$$

Then,

$$t_0 = H_0^{-1} \int_0^\infty \frac{dz}{(1+z)E(z)}. \tag{12}$$

Thus, for example, if we lived in an *Einstein-de Sitter universe* (i.e., $\Omega_m = 1$ and $\Omega_\Lambda = 0$), then $E(z) = (1+z)^{3/2}$ and $t_0 = (2/3)H_0^{-1} \simeq 6.7\,h^{-1}\,\mathrm{Gyr}$. The integrals for $\Omega_m \neq 1$ and $\Omega_\Lambda \neq 0$ are more complicated but are given for some cases in the usual textbooks. Here, we simply note that the ages are shorter for $\Omega_m > 1$ and $\Omega_\Lambda = 0$ and they are larger if $\Omega_\Lambda = 0$ and $\Omega_m < 1$, or if $\Omega_m = 0$ and $\Omega_m + \Omega_\Lambda = 1$ but $\Omega_\Lambda > 0$. Just as a historical aside, a few years ago, it was believed that globular-cluster ages were as high as 15–20 Gyr, older than the age of the Universe for the then-central values of h, Ω_m, and Ω_Λ. Now, however, the current cosmological parameters indicate an age $t_0 \simeq 13.8$ Gyr, consistent more or less with current globular-cluster ages.

4 Redshift of photons

If a light signal is emitted with wavelength λ_e by a comoving object at some time t_e, then when it is observed by a comoving observer at some later time t_o, it will have a wavelength

$$\lambda_o = \lambda_e \frac{a(t_o)}{a(t_e)} \equiv 1 + z.$$

This also defines the *redshift* z. This result follows heuristically by noting that the wavelength of the photon increases with the scale factor of the Universe, but it can be derived formally (as you will do in a homework problem) from the geodesic equation or by identifying conserved quantities associated with Killing vectors of the FRW spacetime. This formula agrees for $z \ll 1$ with our earlier result, where we associated the redshift with the Doppler shift from a recessional velocity, but it also applies to the case where $z \gtrsim 1$. In cosmology, when we talk about a galaxy "at a redshift z", we are referring to a galaxy that we see now when the size of the Universe was a factor $(1+z)^{-1}$ smaller than it is now. The most distant galaxies observed so far have $z \sim 7$. As we will see later, the cosmic microwave background was emitted from $z \simeq 1100$.

With the geodesic equation, it can also be shown that a massive particle emitted by a comoving observer at time t_e with momentum p_e will be seen to have a momentum

$$p_o = p_e \frac{a(t_e)}{a(t_o)} = \frac{1}{1+z},$$

when it passes a comoving observer at time t_o. This follows heuristically by noting that the de Broglie wavelength $\lambda \propto p^{-1}$ increases with $a(t)$.

5 Horizons

Recall the FRW metric in the form,

$$ds^2 = dt^2 - a^2(t) \left[d\chi^2 + \begin{pmatrix} \sin^2\chi \\ \chi^2 \\ \sinh^2\chi \end{pmatrix} (d\theta^2 + \sin^2\theta \, d\phi^2) \right], \tag{13}$$

and also recall that photons move along geodesics, $ds = 0$. Consider a photon emitted at time t_e from the origin ($\chi = 0$) that moves in the direction $\theta = \phi = 0$. Then, from $ds = 0$, it follows that the photon moves along a trajectory with $dt = a d\chi$, and therefore

$$\chi = \int_{t_e}^{t_0} \frac{dt}{a(t)}, \tag{14}$$

is the coordinate distance traveled by the photon between times t_e and t_o. For example, in a flat MD Universe, $a \propto t^{2/3}$ and so the physical distance traveled by the photon is

$$a_0\chi = 3t_0[1 - (t_e/t_0)^{1/3}] = 3t_0[1 - (1+z_e)^{-1/2}]. \tag{15}$$

In such a model, the maximum distance traveled by a photon between time $t = 0$ and today is $a_0 r = 3t_0 = 2/H_0 = 6000\,h^{-1}$ Mpc, where we have written the Hubble constant as $H_0 = 100\,h$ km/sec/Mpc, and, as mentioned above, the numerical value is $h \sim 0.7$.

For our Universe, which has $\Omega_m \simeq 0.3$ and $\Omega_\Lambda \simeq 0.7$, the numerical value for the horizon distance is slightly different. This means that there is a finite observable volume for the Universe.

6 Classical cosmological tests

In the following, we will assume that the Universe consists of nonrelativistic matter and possibly a cosmological constant. Before proceeding, let us justify our neglect of radiation. A variety of measurements (including, for example, simply weighing galaxies and then measuring their space density) indicate a nonrelativistic-matter density $\rho_m \simeq 0.3\rho_c \simeq 1.5 \times 10^{-6}$ GeV/cm^3 (for $h \simeq 0.7$). We observe a cosmic microwave background, a blackbody spectrum of photons, with a temperature of $T_0 \simeq 2.7$ K. From the Stefan-Boltzmann equation, this corresponds to an energy density $\rho_\gamma \simeq 3\times 10^{-10}$ GeV/cm^3. We also have good reason to believe that there is a cosmic neutrino background with an energy density roughly 2/3 that of the photon density. Thus, radiation (neutrinos plus photons) has an energy density only $\sim 3 \times 10^{-4}$ of the matter density today. Since $\rho_{\rm rad} \propto (1+z)^4$ while $\rho_m \propto (1+z)^3$, radiation will be negligible compared with matter as long as we are at redshifts less than $z_{\rm eq} = (\rho_m/\rho_{\rm rad})_0 \simeq 3000$. At earlier times, the Universe was radiation dominated.

6.1 Angular-diameter distance

Heuristically, if the size of an object is known, its distance can be inferred by determining how big it appears to be—i.e., the angle it subtends when we view it. In cosmology, the *angular-size distance* takes into account the effects of expansion and geometry to relate the observed angular size of an object of known proper size to its distance. This is done as follows: The coordinate distance from light emitted at time t_e to t_0 is

$$\chi = \int_{t_e}^{t_0} \frac{dt}{a} = \int_{a_e}^{a_0} \frac{da}{a\dot{a}} = \frac{1}{H_0}\int_{a_e}^{a_0} \frac{da}{a^2 E(z)} = \frac{1}{a_0 H_0}\int_0^{z_e} \frac{dz}{E(z)}, \tag{16}$$

where the function $E(z)$ describes the time evolution of the expansion rate, $H(z) = H_0 E(z)$. From the form of the metric, we know that at time t_e, the circumference of a great circle of coordinate radius χ is $2\pi a(t_e) S_\chi$, where $S_\chi = (\sinh\chi, \chi, \sin\chi)$ for open, flat, and closed universes, respectively. Therefore, if we see today an object of proper size D, then the angle it subtends on the sky is $\theta = D/[a(t_e)S_\chi] \equiv D/d_A$, where $d_A(z_e) = a(t_e)S_\chi$. Thus,

$$d_A(z) = \frac{a_0}{1+z}\left\{\begin{array}{c} \sinh\left[\frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)}\right] \\ \frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)} \\ \sin\left[\frac{1}{a_0 H_0}\int_0^z \frac{dz}{E(z)}\right] \end{array}\right\}, \tag{17}$$

for (from top to bottom) open, flat, and closed universes. For example, in an Einstein-de Sitter universe,

$$d_A(z) = \frac{2H_0^{-1}}{1+z}\left[1 - \frac{1}{\sqrt{1+z}}\right], \tag{18}$$

which, for $z \ll 1$ becomes $d_A \simeq H_0^{-1} z$, indicating that we recover the expected behavior at small distances. You can also show that this linear relation is recovered for any Ω_m or Ω_Λ. It can also be shown that to quadratic order in z, $H_0 d_a(z) \simeq z - (1/2)(3+q_0)z^2 + \cdots$, where q_0 is the deceleration parameter. For future use, it will be convenient to define a scaled distance, $y(z) \equiv H_0(1+z)d_A(z)$. Expressions for $y(z)$ involve more complicated integrals for $\Omega_m \neq 1$ and for $\Omega_\Lambda \neq 0$. For $\Omega_\Lambda = 0$ and $z \gg \Omega_m^{-1}$, $y(z) \simeq 2/\Omega_m$. In practice, it is difficult to find objects (like galaxies) of fixed known size D, making the determination of the angular-diameter distance difficult.

6.2 Luminosity distance

If we know the intrinsic luminosity L of a source, then we can determine its distance by measuring the energy flux F we observe from this source. The *luminosity distance* of a cosmological source is defined by $d_L^2 \equiv L/(4\pi F)$. The flux F we observe at time t_0 from a source at a distance χ is

$$F = \frac{L}{4\pi a^2(t_0) S_\chi^2 (1+z)^2}. \tag{19}$$

This result is arrived at in the following way: If the detector area is dA, the fraction of the 2-sphere, centered on the source, that is covered by the detector is $dA/[4\pi a^2(t_0)S_\chi^2]$. Then there is an additional factor of $1+z$ that is due to the redshift of photon energy, and there is another factor $1+z$ due to the redshift of the emission rate (if the source emits in its rest frame a signal with a period P, it is observed with period $(a_0/a_e)P$. Therefore, $d_L = d_A(1+z)^2$. Note again that to quadratic order, $H_0 d_L(z) = z + (1/2)(1-q_0)z^2 + \cdots$. If there is a "standard candle", a class of objects of fixed known luminosity, then the parameters H_0 and q_0 can be determined by measuring the observed flux as a function of redshift. Thus, $q_0 = \Omega_m/2 - \Omega_\Lambda$ has been obtained by measuring the quadratic correction to the Hubble law with distant (Type Ia) supernovae leading to the value $q_0 \simeq -0.55$ mentioned before.

6.3 Proper displacement

For a number of applications, it is important to know the proper-distance interval dl covered in a redshift interval dz. This is obtained by noting that a light ray covers a distance

$$dl = dt = \frac{da}{\dot{a}} = \frac{dz}{1+z}\frac{a}{\dot{a}}, \tag{20}$$

from which it follows that

$$\frac{dl}{dz} = \frac{H_0^{-1}}{(1+z)E(z)}. \tag{21}$$

6.4 The resolution of Olber's paradox

In a homogeneous, static, and infinite Universe, every line of sight eventually ends up on a galaxy, and if so, then the night sky should be bright. We can now understand why in an expanding Universe the night sky is dark. Consider a population of objects of cross section σ (e.g., $\sim \pi(10\,\mathrm{kpc})^2$ for spiral galaxies) with a constant number per comoving volume and current number density n_0. That means that the proper number density as a function of redshift is $n(z) = n_0(1+z)^3$. The probability that a given line of sight intersects such an object between z and $z + dz$ is

$$\frac{dP}{dz} = \sigma n(z)\frac{dl}{dz} = \sigma n_0 H_0^{-1}\frac{(1+z)^2}{E(z)}. \tag{22}$$

At $z \gg \Omega_m^{-1}$, $E(z) \to \Omega_m^{1/2}(1+z)^{3/2}$, and the optical depth for intersecting a galaxy out to a redshift z is

$$\tau(z) = \int^z dP = \frac{2}{3}\frac{\sigma n_0 c H_0^{-1}}{\Omega_m^{1/2}}(1+z)^{3/2}. \tag{23}$$

Ordinary galaxies have a local number density $n_g \sim 0.02\,h^3\,\mathrm{Mpc}^{-3}$, and radii $r_g \sim 10\,h^{-1}$ kpc, from which we obtain $\tau \sim 0.01\,(1+z)^{3/2}\Omega_m^{-1/2}$. Therefore, out to $z = 1$, about $0.04\,\Omega_m^{-1/2}$ of the sky is covered by galaxies, and full coverage ($\tau = 1$) is reached only at $z \sim 20\,\Omega_m^{1/3}$. Thus, Olber's paradox is explained if galaxies don't form or light up fully until $z \sim$few.

6.5 Number counts

We can also calculate the number of objects seen in a given redshift interval dz in a solid angle $\delta\Omega$ on the sky, under the assumption that the comoving number density of such objects remains constant. The area subtended by an angle $\delta\Omega$ at a redshift z is $\delta A = a^2 S_\chi^2 \delta\Omega$. Then, $(dl/dz)dz$ is the proper linear depth from z to $z + dz$, so the differential volume in the redshift interval dz and solid angle $\delta\Omega$ is

$$\delta V = \frac{H_0^{-1}\delta z}{(1+z)E(z)}\frac{(a_0 S_\chi)^2\delta\Omega}{(1+z)^2}. \tag{24}$$

Using $n(z) = n_0(1+z)^3$, we find that the number of galaxies in dz per steradian on the sky is

$$\frac{d\mathcal{N}}{dz} = n_0 H_0^{-3} F_n(z), \tag{25}$$

where

$$F_n(z) = \frac{[H_0 a_0 S_\chi(z)]^2}{E(z)}. \tag{26}$$

Since S_χ and $E(z)$ depend on the matter content and the geometry, measuring the number counts as a function of redshift can in principle be used to determine cosmological parameters. In practice, though, the abundances of the target populations (e.g., galaxies or clusters of galaxies) undergo evolution in complicated ways, and this evolution is difficult to disentangle from the cosmological effects.

6.6 "Superluminal" proper motions

The black holes that power active galactic nuclei can often emit jets with relativistic velocities. Suppose that such a source emits a jet with velocity v at an angle θ from the line of sight. Then, after a time δt, the jet will have propagated a distance $v\delta t\cos\theta$ toward us and a transverse distance $v\delta t\sin\theta$. Suppose now that the observer-source distance is D_{os}. The observer will see the signal at a transverse distance $\delta l_\perp = v\delta t\sin\theta$ after a time

$$\begin{aligned}\Delta t_{\rm obs}(\delta t) &= \delta t + \left[(D_{os} - v\delta t\cos\theta)^2 + (v\delta t\sin\theta)^2\right]^{1/2} \\ &= \delta t + \left[D_{os}^2 + (v\delta t)^2 - 2D_{os}v\delta t\cos\theta\right]^{1/2} \qquad (27) \\ &\simeq D_{os} + \delta t(1 - v\cos\theta), \qquad (28)\end{aligned}$$

Therefore, the apparent transverse velocity is

$$\frac{\delta l_\perp}{\delta t} = \frac{v\sin\theta}{1 - v\cos\theta}, \tag{29}$$

which is maximized, for a given v, at $\cos\theta = v$. So, $(\delta l_\perp/\delta t)_{\rm max} = \gamma v$, which is faster than the speed of light for $v > c/\sqrt{2}$. If the source is at redshift z, then the observed time interval is $\delta t_o = (1+z)\delta t$, and

$$\delta\theta = \frac{\delta l_\perp}{a(z)S_\chi(z)} = \frac{\delta l_\perp(1+z)}{a_0 S_\chi(z)}. \tag{30}$$

Therefore, the observed angular proper motion is

$$\mu = \frac{d\theta}{dt} = \frac{1}{a_0 S_\chi(z)}\frac{v\sin\theta}{1 - v\cos\theta}, \tag{31}$$

and

$$\mu_{\rm max}(z) = \frac{\gamma v}{a_0 S_\chi(z)}. \tag{32}$$

Since Ω_m and Ω_Λ fix $H_0 a_0 S_\chi(z)$, the maximum proper motion is determined by $H_0\gamma v$. Just for reference, observations indicate $h\gamma \simeq 10$. The idea is then to measure v, θ, and μ (assumed to be less than $\mu_{\rm max}$) to constrain Ω_m and Ω_Λ.