Physics 201: Special Relativity and Waves
Fall Semester, 2000

FINAL EXAM December 18, 2000

Work all SIX problems. This is a closed book exam, but you may use a calculator and one sheet of notes (two sided). Show your work as partial credit will be given. The first parts of multi-part problems are generally easier than the later parts, so be sure to attempt each problem. The maximum possible score is 200 points. The point value of each problem is as indicated.

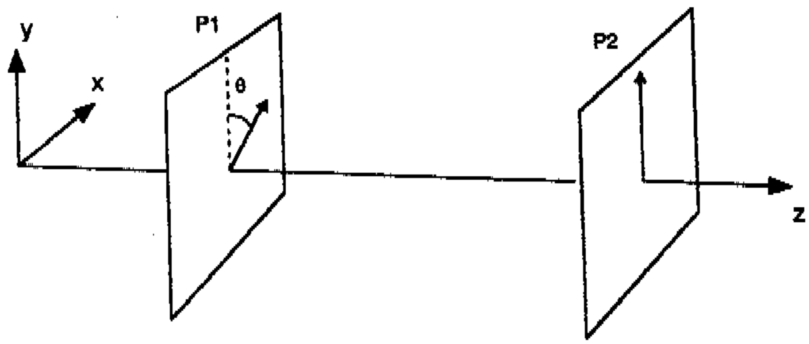
Graded exams and course grades may be obtained on or after December 20 from Mr. Lonnie Clark in Blmbg 327. Grades may also be obtained by sending email to D. Reich (dhr@pha.jhu.edu).

Problem 1 (40 points)

(a) Consider a rectangular drumhead with dimensions $L \times 2L$ with all four sides held fixed. If the phase velocity for the drumhead is denoted by v, find the 4 lowest normal mode frequencies for the drumhead. Express your answer in terms of L and v.

(b) Now suppose that one of the short sides (of length L) is free, while the other three sides remain fixed. Find the 4 lowest mode frequencies in this case.

Problem 2 (30 points)



(a) Plane waves of light, initially linearly polarized along the y-axis, travel along the z-axis in the figure above, and shine through two perfect pieces of polaroid P1 and P2. P1 is oriented so that its easy-transmission axis makes an angle $\theta = 30°$ with the $\hat{y}$ axis. P2 has its easy-transmission axis along $\hat{y}$. Neglecting losses due to reflections, what is the ratio of the transmitted intensity to the incident intensity?

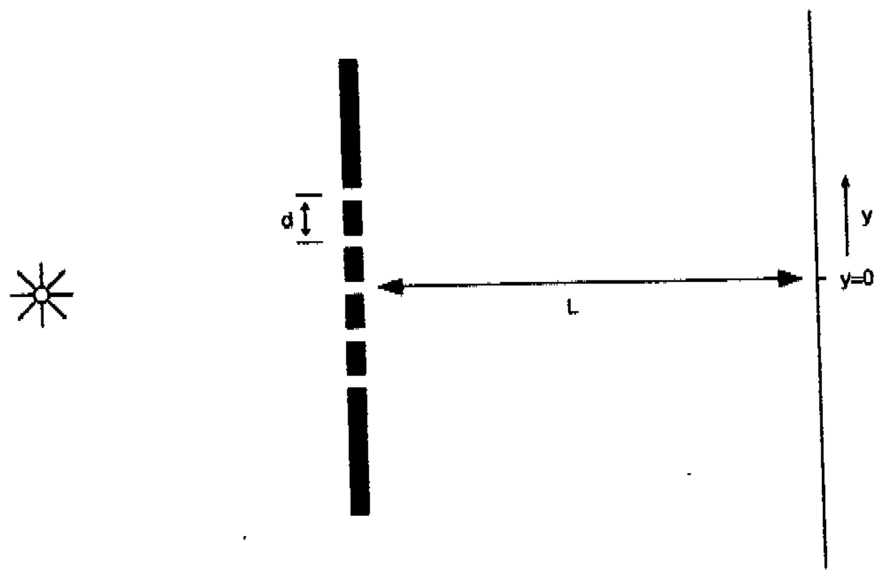
15 pts

(b) The linearly polarized light source in part (a) above is replaced by a circularly polarized light source. Again, calculate the ratio of the transmitted intensity to the incident intensity. In addition, describe the polarization state of the transmitted wave.

15 pts

Problem 3: (40 points)

Monochromatic light of wavelength $\lambda = 600$ nm from a distant point source illuminates an opaque screen with 5 identical, narrow slits. The separation between neighboring slits is $d = 0.2$ mm. The light transmitted through these slits produces an interference pattern on a second screen at a distance $L = 2$ m from the first screen. In this problem you may neglect single-slit diffraction effects.



(a) Find the distance Δy between successive *principal* maxima of the interference pattern.

(b) Find the distance $\Delta y'$ between successive minima in the interference pattern.

(c) Now the monochromatic source is replaced by a second, coherent light source that produces light of two wavelengths, $\lambda_1 = 600$ nm, and $\lambda_2 = 610$ nm. How far in y (distance along the second screen) must you go before the two components of the light are "resolved" by the slits? Here, "resolved" means that a principal maximum from the light with wavelength λ_2 falls on top of an intensity minimum of the light of wavelength λ_1.

Problem 4 (20 points)

A stick is initially stationary in Frame $\mathcal{O}$. It has rest length L_0 and makes an angle θ_0 with the x-axis. It is then made to move relative to Frame $\mathcal{O}$ at a constant velocity v parallel to the x-axis. Find the length L of the stick as measured by an observer who stays at rest in Frame $\mathcal{O}$.

Problem 5 (40 points)

Deep in a southern swamp, a relativistic game of Capture the Flag takes place. Two trains are moving toward each other on parallel tracks. Observers on one train measure the velocity of the other train to be $v = 0.6c$. Train 1 has rest length $L_1 = 110$ m, and Train 2 has rest length $L_2 = 100$ m. Observer A sits at the front of Train 1, and Observer W sits at the front of Train 2. The object of the game is to be the first to grab a flag hanging at the back of the other observer's train.

After the flags have been grabbed, A says, "Aha! My train is longer. Thus I reached the back of W's train and grabbed his flag before he could reach the back of mine. Therefore I am the winner!"

"No, that's fuzzy math." replies W, "My advisors who rode on my train with me say that your train was Lorentz contracted, and was in fact shorter than mine. Therefore, I got to the back of your train first, and thus I won!"

"Oh, yeah?" responds A, "Well, Lorentz contraction works for me too, and so your train must have been even shorter than I thought it was at first. I still win!"

(a) Analyze the time-ordering of the flag-grabbing events, and assess the validity of each observer's claim.

(b) "Gee," says A, "maybe if my train had been longer, everyone would have agreed that I won." Determine whether A could have made his train long enough that **all** observers would agree that he won the game, and give the minimum length needed if he could have done so.

Problem 6 (30 points)

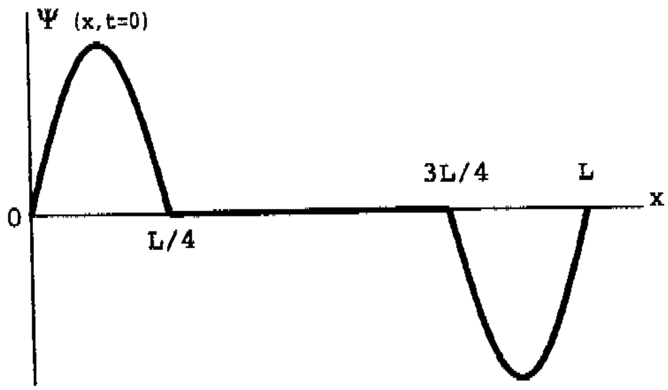
A string with linear mass density ρ and length L is stretched under equilibrium tension T with its ends fixed. At time $t = 0$ the string is given a displacement

$$\Psi(x, t=0) = \begin{cases} A \sin 4\pi x/L & \text{if } 0 < x < L/4 \\ 0 & \text{if } L/4 < x < 3L/4 \\ A \sin 4\pi x/L & \text{if } 3L/4 < x < L \end{cases} \quad (1)$$

as shown below.

+2

Correct ~~sign~~ dots two



B_2 / B_4

$\frac{4A}{3\pi}$

$\frac{A}{2}$

15 (a) If the string is released from rest, find the frequencies of the two lowest-frequency normal modes that have non-zero amplitude.

15 (b) Determine the amplitude of those two normal modes.

Here follows a smattering of indefinite integrals:

$$\int (\sin x)^2 dx = \frac{x}{2} - \frac{\sin 2x}{4} + C \quad (2)$$

$$\int \sin(nx)\sin(mx)dx = \frac{\sin((n-m)x)}{2(n-m)} - \frac{\sin((n+m)x)}{2(n+m)} + C \quad (3)$$

$$\int \sin x \cos x dx = -\frac{x}{2}(\cos x)^2 + C \quad (4)$$

$$\int \sin(nx)\cos(mx)dx = \frac{\cos((n-m)x)}{2(n-m)} - \frac{\cos((n+m)x)}{2(n+m)} + C \quad (5)$$

$$\int (\cos x)^2 dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \quad (6)$$

$$\int \cos(nx)\cos(mx)dx = \frac{\sin((n-m)x)}{2(n-m)} + \frac{\sin((n+m)x)}{2(n+m)} + C \quad (7)$$

DHR

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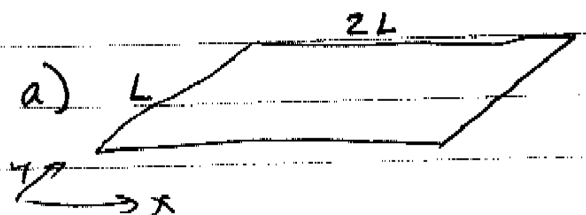
Solutions

1 a)



Has solutions $\psi_{nm}(x,y,t) \propto \sin(k_x x)\sin(k_y y)\, e^{i\omega t}$

with $\omega^2 = v^2(k_x^2 + k_y^2)$

$$k_x = \frac{n\pi}{2L} \qquad k_y = \frac{m\pi}{L}$$

$$\omega^2 = v^2\left(\left(\frac{n\pi}{2L}\right)^2 + \left(\frac{m\pi}{L}\right)^2\right)$$

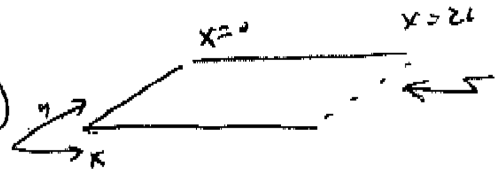
$$\omega = \frac{v\pi}{L}\left(\frac{n^2}{4} + m^2\right)^{1/2}$$

table of $\frac{L\omega}{v\pi}$:

n \ m	1	2	3
1	$\sqrt{5/4}$	$\sqrt{17/4}$	
2	$\sqrt{8/4}$	$\sqrt{20/4}$	
3	$\sqrt{13/4}$		
4	$\sqrt{20/4}$		

4 lowest modes

n	m	ω
1	1	$\frac{v\pi}{L}\sqrt{5/4} = 3.51\ v/L$
2	1	$\frac{v\pi}{L}\sqrt{2} = 4.44\ v/L$
3	1	$\frac{v\pi}{L}\sqrt{13/4} = 5.66\ v/L$
1	2	$\frac{v\pi}{L}\sqrt{\frac{17}{4}} = 6.48\ v/L$

(1) (b) 

$\leftarrow$ Free edge.

Now, re-examine differential eqn:

$$\frac{1}{f} f''(x) + \frac{1}{g} g''(y) + \frac{\omega^2}{v^2} = 0$$

$$f'' + k_x^2 f = 0 \quad \Rightarrow \quad f(x) = A\cos k_x x + B\sin k_x x$$

$f(x=0) = 0 \Rightarrow A = 0$

free edge at $x = 2L \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=2L} = 0$

i.e. $\left. k_x B \cos k_x x \right|_{x=2L} = 0$

$$\Rightarrow k_x \cdot 2L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots = \pi\left(\tfrac{1}{2} + n\right) \quad n = 0, 1, 2 \ldots$$

$$k_x = \frac{\pi}{2L}\left(\frac{2n+1}{2}\right) = \frac{\pi(2n+1)}{4L}$$

$$\omega = V\sqrt{k_x^2 + k_y^2}$$

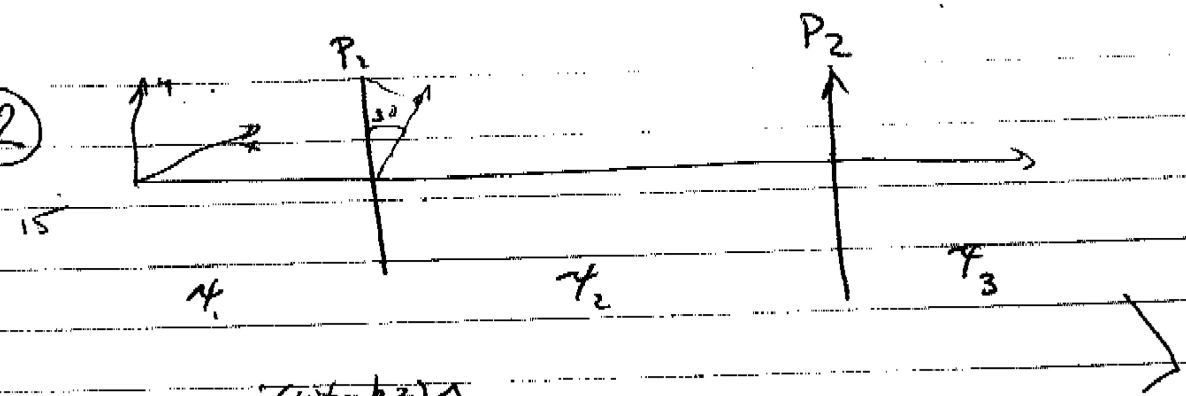
$$= V\left(\frac{\pi^2}{16L^2}(2n+1)^2 + \frac{\pi^2 m^2}{L^2}\right)^{1/2} = \frac{V\pi}{4L}\left((2n+1)^2 + 16m^2\right)^{1/2}$$

Table of $\frac{4L\omega}{V\pi}$

n \ m	1	2
0	$\sqrt{17}$	$\sqrt{65}$
1	$\sqrt{25}$	$\sqrt{73}$
2	$\sqrt{41}$	
3	$\sqrt{65}$	

n	m	ω
0	1	$\frac{V\pi}{4L}\sqrt{17} = 3.24\ V/L$
1	1	$\frac{V\pi}{4L} 5 = 3.92\ V/L$
2	1	$\frac{V\pi}{4L}\sqrt{41} = 5.03\ V/L$
3	1	$\frac{V\pi}{4L}\sqrt{65} = 6.33\ V/L$
0	2	

(2)



$\psi_1 = A e^{i(\omega t - kz)} \hat{y}$

Direction of P_1 easy axis $\hat{P}_1 = \hat{x}\sin\theta + \hat{y}\cos\theta$

$\psi_2 = A e^{i(\omega t - kz)} \cos\theta \, \hat{P}_1$

$= A e^{i(\omega t - kz)} (\cos\theta \sin\theta \, \hat{x} + \cos^2\theta \, \hat{y})$

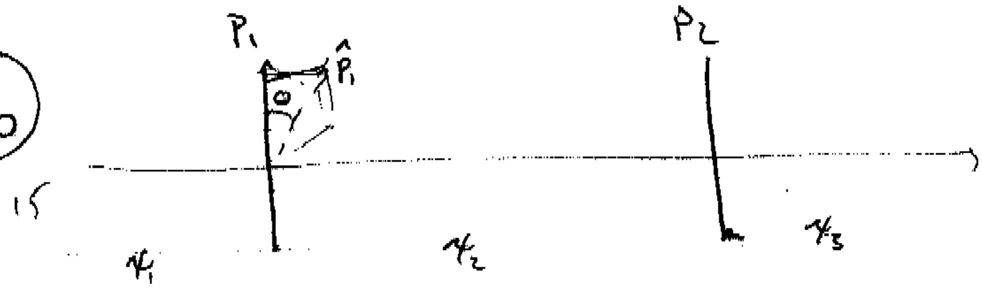
P_2 picks out $\hat{y}$ component $\Rightarrow$

$\psi_3 = A e^{i(\omega t - kz)} \cos^2\theta \, \hat{y}$

Intensity : $\dfrac{I_3}{I_1} = \dfrac{\psi_3 \psi_3^*}{\psi_1 \psi_1^*} = \dfrac{A^2 \cos^4\theta}{A^2} = \cos^4\theta$

$= \cos^4(30°) = \left(\frac{\sqrt{3}}{2}\right)^4 = \boxed{9/16 = I_3/I_1}$

2 b



$$\psi_1 = A\psi_+ = A e^{i(\omega t - kz)}\left(\frac{\hat{x} + i\hat{y}}{\sqrt{2}}\right)$$

Intensity: $I_1 = \psi_1 \psi_1^* = A^2$

$$\psi_2 = \frac{A e^{i(\omega t - kz)}}{\sqrt{2}}\left(\sin\theta\, \hat{p} + i\cos\theta\, \hat{p}\right)$$

$$= \frac{A e^{i(\omega t - kz)}}{\sqrt{2}}\left(\cancel{\sin\theta(\sin\theta\,\hat{x} + \cos\theta} \;\; (\sin\theta + i\cos\theta)(\hat{x}\sin\theta + \hat{y}\cos\theta)\right)$$

Again P_2 picks out $\hat{y}$ compnt

$$\psi_3 = \frac{A e^{i(\omega t - kz)}}{\sqrt{2}}\cos\theta(\sin\theta + i\cos\theta)\hat{y}$$

Intensity

$$I_3 = \psi_3\psi_3^* = \frac{A^2}{2}\cos^2\theta$$

$$\frac{I_3}{I_1} = \frac{\cos^2\theta}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 = \boxed{3/8 = \frac{I_3}{I_1}}$$

(3) use $I = \left\{ \dfrac{\sin(Nkd\sin\theta/2)}{\sin(kd\sin\theta/2)} \right\}^2$

here, $\sin\theta \sim \tan\theta = y/L$

$k = 2\pi/\lambda$

$$I = \left\{ \frac{\sin(N\pi dy/L\lambda)}{\sin(\pi dy/L\lambda)} \right\}^2$$

a) Principal maxima occur when denominator $= 0$

ie $\pi dy/L\lambda = n\pi$

$y = L\lambda n/d$

$$\Delta y = y(n+1) - y(n) = \frac{L\lambda}{d} = \frac{2m \times 6\times10^{-7}m}{2\times10^{-4}m} = 6\times10^{-3}\,m$$

$\Delta y = 6mm$

b) minima occur when Numerator $= 0$

ie $N\pi dy/L\lambda = m\pi$

$$y = \frac{L\lambda}{Nd}m$$

$$\Delta y' = y(m+1) - y(m) = \frac{L\lambda}{Nd} = \frac{\Delta y}{N} = \frac{6mm}{5}$$

$\Delta y' = 1.2\ mm$

③c) for given n, locations of principal maxima for different wavelengths:

$$y_1 = \frac{L\lambda_1 n}{d}$$

$$y_2 = \frac{L\lambda_2}{d} n$$

"Resolution" condition $\Rightarrow$ $y_2 - y_1 = \Delta y'$

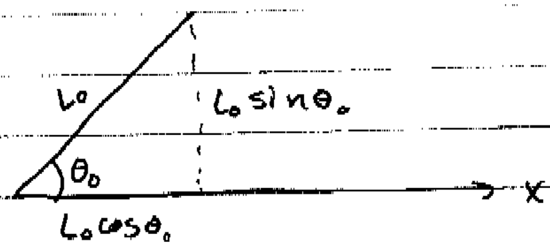
$$\frac{L\lambda_2 n}{d} - \frac{L\lambda_1 n}{d} = \frac{L\lambda_1}{Nd}$$

$$n = \frac{\lambda_1}{N(\lambda_2 - \lambda_1)} = \frac{600\,nm}{5 \cdot 10\,nm} = 12$$

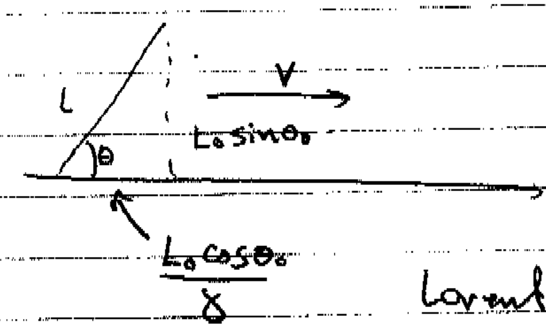
$$n = 12$$

$\Rightarrow$ distance $y = n\Delta y = 12 \cdot 6\,mm$
$= 72\,mm$ //

4



(units where $c=1$)



$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

Lorentz contraction along direction of motion only

$$L = \left\{ \frac{L_0^2 \cos^2\theta_0}{\gamma^2} + L_0^2 \sin^2\theta_0 \right\}^{1/2}$$

$$= L_0 \left\{ \cos^2\theta_0 \left(\frac{1}{\gamma^2} + 1 - 1 \right) + \sin^2\theta_0 \right\}^{1/2}$$

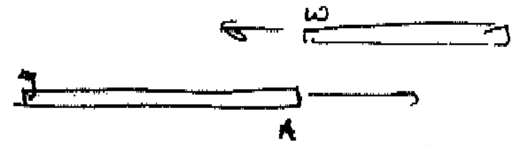
$$= L_0 \left\{ 1 - \cos^2\theta_0 \left(1 - \frac{1}{\gamma^2} \right) \right\}^{1/2}$$

$$1 - 1/\gamma^2 = 1 - (1 - v^2) = v^2$$

$$L = L_0 \left(1 - v^2 \cos^2\theta_0 \right)^{1/2}$$

$$(0 \leq |v| < 1)$$

(5)



Event: 1 front of trains pass = (origin)
2 A grabs W's flag
3 W grabs A's flag

Event	A: t	A: x	W: t'	W: x'
1	0	0	0	0
2	t_2	0	L_2/v	$-L_2$
3	L_1/v	$-L_1$	t_3'	0

$$\gamma = \frac{1}{\sqrt{1-v^2}} = 5/4$$

$$t_2 = \frac{L_2/\gamma}{v} = L_2/\gamma v$$

$$t_3' = \frac{L_1/\gamma}{v} = L_1/\gamma v$$

$$t_2 = \frac{L_2}{\gamma v} = \frac{100 \text{ m}}{5/4 \cdot 3/5} = 133 \text{ lm}$$

$$t_3 = \frac{L_1}{v} = \frac{110}{3/5} = 183 \text{ lm}$$

$t_2 < t_3 \Rightarrow$ A thinks he's won!

5 a) cont

$t_2' = \frac{L_2}{v} = \frac{100}{3/5} = 167 \text{ lm}$

$t_3' = \frac{L_1}{\gamma v} = \frac{110}{5/4 \cdot 3/5} = 146 \text{ lm}$

$t_3' < t_2' \Rightarrow$ W thinks he's won also!

Reason: Events 2 & 3 are separated by a space like interval

$\Delta\tau^2 = (\Delta t_{23}^2 - \Delta x_{23}^2)$

$= (t_2 - t_3)^2 - (x_2 - x_3)^2$

$= (50 \text{ lm})^2 - (110 \text{ lm})^2 = -9600 \text{ lm}$

& time-ordering is ambiguous for space like intervals

$\Rightarrow$ No clear winner!

(5 b) Need to look at interval as function of L_1. If it can be made time like, then all observers agree on time-ordering.

$$\Delta\tau^2 = (t_2 - t_3)^2 - (x_2 - x_3)^2$$

$$= \left(\frac{L_2}{\gamma v} - \frac{L_1}{v}\right)^2 - (L_1)^2$$

$$= \frac{L_2^2}{\gamma^2 v^2} - \frac{2L_1L_2}{\gamma v^2} + \frac{L_1^2}{v^2} - L_1^2$$

$$= \frac{1}{\gamma^2 v^2}\left(L_2^2 - 2\gamma L_1L_2 + L_1^2(\gamma^2 - v^2\gamma^2)\right)$$

$$= \frac{1}{\gamma^2 v^2}\left(L_2^2 - 2\gamma L_1L_2 + L_1^2\gamma^2(1-v^2)\right)$$

$$\Delta\tau^2 = \frac{1}{\gamma^2 v^2}\left(L_2^2 - 2\gamma L_1L_2 + L_1^2\right)$$

for fixed γ & L_2, L_1 can always be made big enough s.t. L_1^2 term dominates & $\Delta\tau^2 > 0$

To find minimum L_1: Set $\Delta\tau^2 = 0$

$$\Rightarrow L_2^2 - 2\gamma L_1L_2 + L_1^2 = 0$$

$$L_1 = \frac{2\gamma L_2 \pm \sqrt{4\gamma^2 L_2^2 - 4L_2^2}}{2}$$

$$= L_2\left(\gamma \pm \sqrt{\gamma^2 - 1}\right)$$

→

with $\gamma = 5/4$

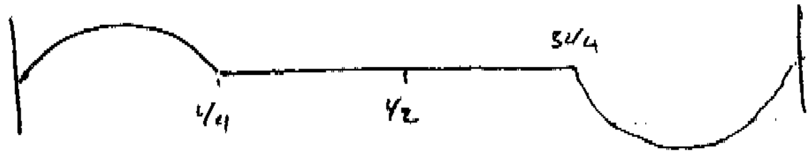
$$L_1 = L_2\left(\frac{5}{4} \pm \frac{3}{4}\right) = 2L_2, \; L_2/2$$

i.e. if $\left.\begin{array}{l} L_1 > 2L_2 \\ L_1 < \frac{1}{2}L_2 \end{array}\right\} \Delta t_c > 0$

Observer A wants $L_1 > 2L_2$

as then he wins in all frames.

6



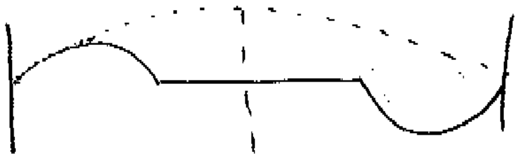
want to expand $\psi(x, t=0)$ in Fourier sine series:

$$\psi(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

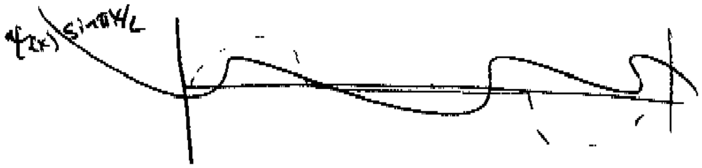
$$B_n = \frac{2}{L}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

a) Try to find which modes will contribute without doing integrals explicity:

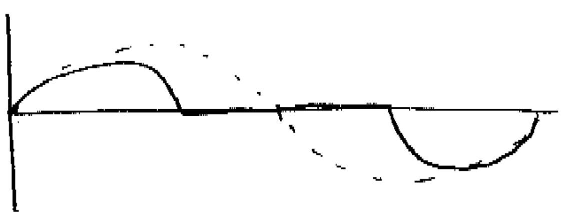
$n=1$



by symmetry $B_1 = 0$

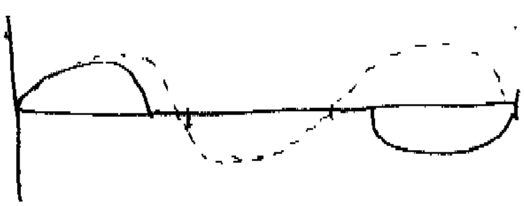


$n=2$



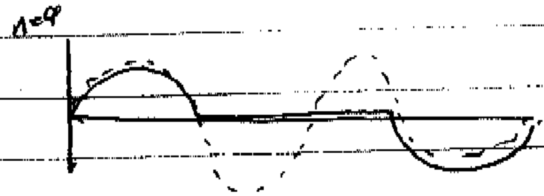
$B_2 \neq 0$

$n=3$



$B_3 = 0$

6 a) cont



$B_4 \neq 0$

So $n = 2, 4$ are lowest modes with non-zero amplitude

~~w/ $\omega = vk = \sqrt{\frac{T}{\rho}}\frac{n\pi}{L}$~~

$\omega = \sqrt{\frac{T}{\rho}}\frac{2\pi}{L}$, $\omega = \sqrt{T/\rho}\,\frac{4\pi}{L}$

b) $B_2 = \frac{2}{L}\int_0^L \psi(x) \sin\frac{2\pi x}{L}\,dx$

$= \frac{2}{L}\int_0^{L/4} A\sin\frac{4\pi x}{L}\sin\frac{2\pi x}{L}\,dx + \frac{2}{L}A\int_{3L/4}^{L}\sin\frac{4\pi x}{L}\sin\frac{2\pi x}{L}\,dx$

equal by symmetry

$= \frac{4A}{L}\int_0^{L/4}\sin\frac{4\pi x}{L}\sin\frac{2\pi x}{L}\,dx$ let $u = \frac{2\pi x}{L}$, $dx = \frac{L\,du}{2\pi}$

$= \frac{4A}{L}\frac{L}{2\pi}\int_0^{\pi/2}\sin 2u\,\sin u\,du$

(see table provided)

$= \frac{4A}{2\pi}\left[\frac{\sin u}{2} - \frac{\sin 3u}{6}\right]_0^{\pi/2}$

$= \frac{2A}{\pi}\left(1/2 - \frac{-1}{6}\right) = \frac{2A}{\pi}\left(1/2 + 1/6\right)$

$= \frac{2A}{\pi}\left(2/3\right) = \frac{4A}{3\pi}$

6 b) cont

$$B_4 = \frac{2}{L}\int_0^L \psi(x)\sin\frac{4\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^{L/4} A\sin^2\frac{4\pi x}{L}\,dx + \frac{2}{L}\int_{3L/4}^{L} A\sin^2\frac{4\pi x}{L}\,dx$$

$$= \frac{4A}{L}\int_0^{L/4}\sin^2\frac{4\pi x}{L}\,dx \qquad u = \frac{4\pi x}{L} \qquad dx = du\,\frac{L}{4\pi}$$

$$= \frac{A}{\pi}\int_0^{\pi}\sin^2 u\,du = \frac{A}{\pi}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Big|_0^{\pi}$$

$$= \frac{A}{\pi}\left(\frac{\pi}{2}\right) = \boxed{A/2 = B_4}$$