kPhysics 171.201
Final Exam

December 17th, 2002

Answer all **eight** problems. Be sure that you pace yourself so that you have enough time to work on each question. Partial credit will be given for partially correct answers. Be sure to **show your working,** and please circle your final answers.

Please use a separate blue book for each section. Identify yourself (on both blue books) by the last 4 digits of your SSN only.

Section A

A1. (30 points) A thin rod of "proper length" L_0 (i.e. length L_0 in its own rest frame) slides along the surface of a photographic plate at speed v = 0.8 c in a direction along its length. A light flash, of negligible duration, illuminates the plate at normal incidence. How long is the photograph of the shadow according to an observer who moves with the rod? How does the observer reconcile this with the known length of the rod? (Give both qualitative and quantitative explanations.)

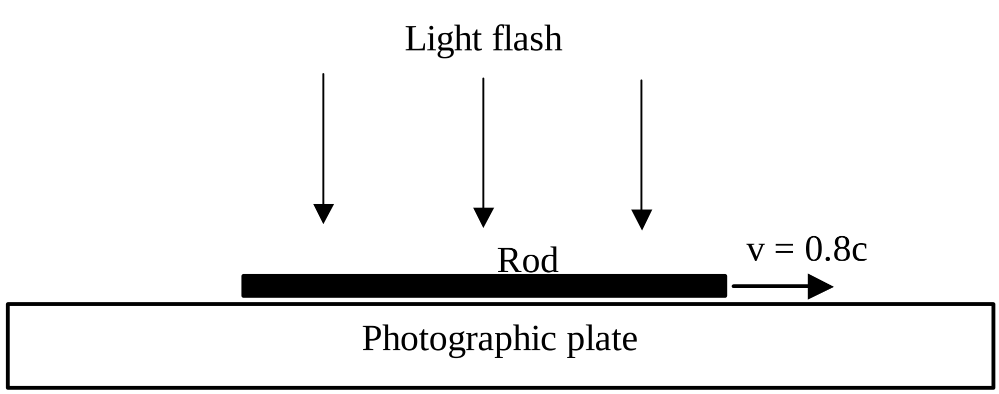


Figure in lab frame (rest frame of photographic plate)

Lorentz factor ? = $(1 - v^2/c^2)^{-1/2}$ = 5/3

Rod is Lorentz contracted in lab frame ✍ photographic image of shadow has proper length L_0/? = 0.6 L_0

In frame of rod, this is further Lorentz contracted to L_0/?^{2} = 0.36 L_0

Explanation: Light flash reaches left and right ends of rod simultaneously in lab frame (normal incidence), but arrival is not simultaneous in rod frame. (This can also be described as the effect of aberration: the light is at normal incidence in the lab frame but not in the rod frame). In rod frame, front (right) end "photographed" first and rear (left) end a time ?t' later. Plate has meanwhile moved to the left a distance v ?t', so the length of the image (in the rod frame) is decreased from L_0 to L_0 – v?t'.

Quantitative:

In the lab frame, the flash reaches the left and right ends of the rod the same time t = ? (t' + vx'/c^2). In the rod frame, the time difference ?t' is therefore –v?x'/c^2 = –vL_0/c^2. The length of the shadow's image in the rod frame is therefore L_0 – v?t' = L_0 – v^2L_0/c^2 = L_0 $(1 - v^2/c^2)$ = L_0/?^{2} as required.

A2. (20 points) Ultra-high energy gamma rays are not detectable from distant galaxies because they are absorbed in intergalactic space as a result of interaction with low-energy "microwave background" photons that pervade the Universe. This interaction leads to the destruction of two photons and the creation of an electron-positron pair. Above what gamma ray energy is this effect important? You may assume the typical energy of a microwave photon is 5×10^{-4} eV, and take the rest-mass energy of the electron as 0.511 MeV.

Suppose the gamma-ray (with energy E_1) is traveling in the +x direction and the microwave photon (with energy E_2) in the –x direction.

Initial 4-momentum = $(E_1/c, E_1/c, 0, 0) + (E_2/c, -E_2/c, 0, 0)$
$= ([E_1+E_2]/c, [E_1-E_2]/c, 0, 0)$

$|\text{ Initial 4-momentum }|^2 = ([E_1+E_2]/c)^2 - ([E_1-E_2]/c) = 4E_1E_2/c^2$

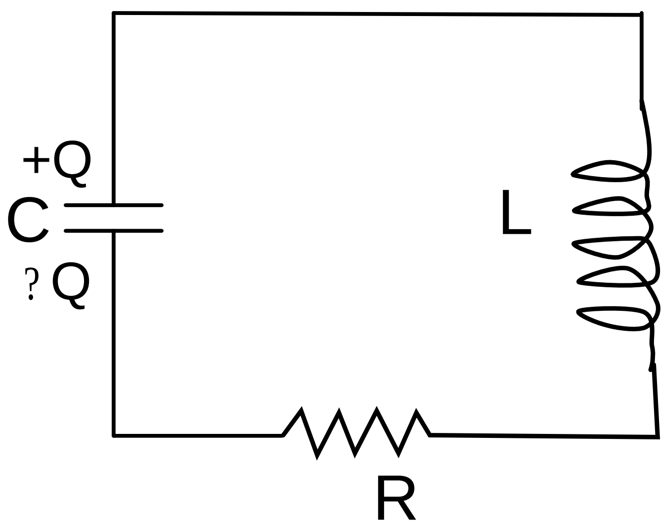
By conservation of 4-momentum, this is also $|\text{ Final 4-momentum }|^2$

An electron-positron pair with no k.e. has 4-momentum $(2m_ec, 0, 0, 0)$ is the center of mass frame ✍ minimum required $|\text{ 4-momentum }|^2 = 4\, m_e^2c^2$

This destruction process can only occur for gamma rays for which $4E_1E_2/c^2 > 4m_e^2c^2$

✍ $E_1 > (m_ec^2)^2/E_2 = [(0.511 \times 10^6)^2 / 5 \times 10^{-4}]\text{ eV} = 5 \times 10^{14}\text{ eV} = 500\text{ TeV}$

A3. (30 points) Consider the circuit shown below, which contains a resistor, an inductor and a capacitor. At time $t = 0$, the capacitor has charge $Q = Q_0$ and the current is zero.



(a) Write down a differential equation for the charge Q(t).

$Ld^2Q/dt^2 + RdQ/dt + Q/C = 0$

(these terms representing respectively the voltage drops across the inductor, resistor and capacitor)

(b) Show (by direct substitution or otherwise) that for sufficiently small resistance R, the solution can be written in the form

$Q = Q_0$ exp (?½? t) cos (? t)

Obtain expressions for ? and ? in terms of R, L and C.

This form implies RdQ/dt = RQ_0 exp (?½? t) (?½? cos (? t) – ? ?sin (? t))

And

$Ld^2Q/dt^2 = LQ_0$ exp (?½? t) (¼ ?$^{?}$cos (? t) + ½? ? ?sin (? t) + ½? ? ?sin (? t) – ?$^{?}$?cos (? t))

Substituting into the D.E., we obtain

[(¼?$^{?}$– ?$^{?}$) L – ½ ?R + 1/C] exp (?½? t) cos (? t) + [? ? L – ? R] exp (?½? t) sin (? t) = 0

This can only be satisfied for all t if both terms in square brackets are zero.

[? ? L – ? R] = 0 ✍ ? ?= R/L

[(¼?$^{?}$– ?$^{?}$) L – ½ ?R + 1/C] = 0

✍ ?$^{?}$ = ¼?$^{?}$– ½ ?R/L + 1/[LC]
= 1/[LC] + ¼?$^{?}$– ½?$^{?}$

✍ ? = [1/(LC) – ?R/2L?$^{?}$]$^{???}$

For given values of L and C, what is the minimum value of R for which this solution applies?

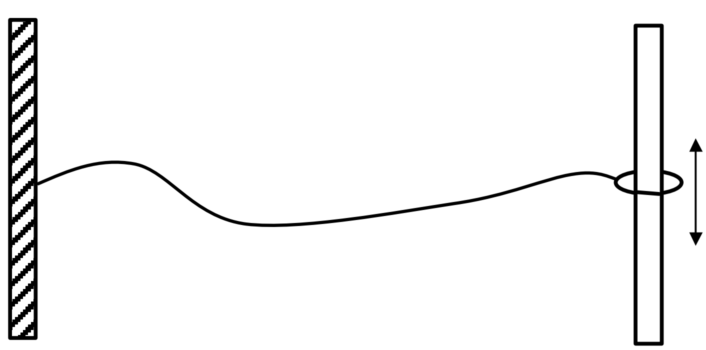
Positive ?$^{?}$ requires 1/(LC) > ?R/2L?$^{?}$ ✍ R > 2(L/C)$^{???}$

(c) Obtain an expression for the instantaneous rate of power dissipation in the resistor. In the limit ? << ? , roughly what fraction of the total energy is lost in a single oscillation period, ???? ? Show that your result consistent with the rate of decline in the amplitude of the oscillation.

Rate of ohmic power dissipation = $I^2 R = R (dQ/dt)^2$
= RQ_0^2 exp (? ? t) [?½? $^?$cos (? t) – ? ?sin (? t)]^2
~ RQ_0^2 exp (? ? t) ? ^{2}?sin^2 (? t) in the limit ? << ? ?
?
Time average power dissipation for complete period ~ ½ ? ^{2} RQ_0^2 exp (? ? t)
Total energy ~ ½ Q_0^2 exp (? ? t) / C
✍ Fraction of energy lost per unit time = ? ^{2} C R = R/L = ?
✍ Fraction of energy lost per period = ??? ?? ?

Consistent with decline in energy in proportion to exp (? ? t) for which dlnE/dt = ? ? ?

A4. (20 points) A string with mass density $?_0$ = 0.02 kg/m and tension T_0 = 0.18 N (18,000 dynes) is fixed at one end (z = 0) and attached to a massless ring that slides without friction on a post at z = L = 3 m. Find the frequencies and wavelengths of the three lowest-frequency normal modes.



Phase velocity: $v_?$ = $(T_0/?_0)^{1/2}$ = (0.18 N / 0.02 kg/m $)^{1/2}$ = 3m/s

Boundary conditions:

?? ??? = 0 at z = 0 (anchored)
(2) d? /dz = 0 at z = L (tension force has to be horizontal)

Solution: ? = A sin (kz) cos (? t)

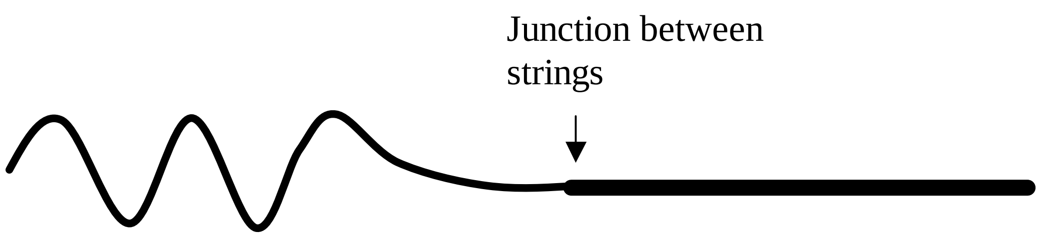
B.C. (1) is satisfied automatically
B.C. (2) requires Ak cos(kL) cos (? t) = 0
✍ kL = ?/2, 3?/2, 5?/2…. ✍ k = (n+½)? ?L

Wavelength, ? = 2?/k = 2L/(n+½) = 4L, 4L/3, 4L/5 ... = 12m, 4m, 2.4m …

Frequency = $v_{??}$?? ?? 0.25 Hz, 0.75 Hz, 1.25 Hz

Section B

B1. (30 points) A sinusoidal traveling wave of wavelength ? travels to the right along a string of mass-per-unit-length $?_0$. The string is attached to a second string (see figure below) which is under identical tension but which has a mass-per-unit-length of 1.21 $?_0$



(a) Compute the reflection and transmission coefficients for the junction between the strings.

> Characteristic impedance for left string Z_L = ?($?_0T_0$)
> Characteristic impedance for right string Z_R= ?(1.21 $?_0T_0$) = 1.1 Z_L
> Reflection coefficient R = $(Z_L - Z_R) / (Z_L + Z_R)$ = (1 – 1.1) / (1 + 1.1)
> = – 0.1/2.1 = – 0.048
> Transmission coefficient T = 1 + R = 0.952

(b) What fraction of the incident power is reflected, and what fraction transmitted.

> Fraction of power reflected = R^2 = 0.0023
> Fraction transmitted = 1 – R^2 = 0.9977

(c) Describe how the reflection amplitude could be made nearly zero by inserting a third length of string between the original two strings. What length and mass-per-unit-length should the third length of string possess?

> Insert a middle piece of string of length ?/4 with an impedance Z_M which makes the two junctions have equal reflection coefficients. This requires $Z_L/Z_M = Z_M/Z_R$ ✍ Z_M = ? (Z_R Z_L)
>
> The tension must be the same in all three strings, requiring that the middle string have a mass-per-unit length of ? ($?_R$ $?_L$) = ? ($?_0$ x 1.21$?_0$) = 1.1⁇$_0$

B2. (25 points) A rectangular waveguide has its long axis along the z–direction. Its dimensions in the x– and y–directions are respectively 2 cm and 3 cm, and its walls are perfectly conducting.

Linearly polarized waves, with the electric field along the x–direction, propagate along z–axis with the electric field varying according to
E_x = A sin (k_yy) cos (? t – k_zz)

(a) What values of k_y are permitted by the boundary conditions

Boundary conditions require E_x = 0 on the walls with y=constant (since transverse E-field must vanish there)

✍ sin (k_yb) = 0 where b = 3 cm is the width in the y–direction
✍ k_y = n?/b with n a positive integer

(b) What is the minimum frequency, ? = ? ?2?, for which waves can propagate along the waveguide. [c = 3 x 10^8m/s = 3 x 10^{10} cm/s]

Dispersion relation is $?^2 = c^2 (k_y^2 + k_z^2)$

Wave propagation requires $k_z^2 > 0$ ✍ ? > c k_y > c?/b
Frequency ? = ? ?2? ?> c/2b = (3 x 10^{10}/6) Hz = 5 GHz

(c) Repeat part (b) for waves polarized with the electric field in the y–direction

Replace b by a = 2cm ✍ Minimum frequency = 7.5 GHz

(d) Compute the phase velocity, ? /k_z , and group velocity, d? /dk_z , for waves of frequency twice the low-frequency cutoff.

Dispersion relation: $?^2 = c^2 (k_{x,y}^2 + k_z^2) = ?_{min}^2 + c^2 k_z^2$
Phase velocity, ? /k_z = c /(1 – $?_{min}^2$?? $^2)^{1/2}$ = $(4/3)^{1/2}$c
Group velocity, d? /dk_z = c^2 k_z?? ?? ?= $(3/4)^{1/2}$c

B3. (30 points)

Two ideal linear polarizers are placed in an initially unpolarized light beam with their transmission axes exactly "crossed": i.e. the second polarizer has its transmission axis perpendicular to that of the first polarizer.

(a) If the incident light beam has energy flux F_0, what is the remaining flux after it passes through the first polarizer? What about after it passes through the second polarizer?

Malus's Law: Flux drops by factor $\cos^2$? ?in passing through any polarizer, where is the angle between the polarizer transmission axis and the E-field of the radiation?

??

Unpolarized radiation has randomly varying polarization ✍ average value of $\cos^2$? ?is ½ ✍ Flux decreased to ½F_0 after first polarizer.

Radiation is now 100% linearly polarized along the axis of the 1^{st} polarizer (call it the y axis). Second polarizer makes an angle ?/2 to the y–axis ✍ $\cos^2$? ?? ?? ?✍ flux decreased to 0 after second polarizer.?

(b) Suppose now a third linear polarizer is placed between the other two polarizers, with its transmission axis at angle ? to that of the first polarizer. What is the emergent flux after the light has passed through all three polarizers? What is the maximum possible emergent flux, and for what ? is it obtained?

Flux is reduced at middle polarizer by a factor $\cos^2$? ?to ½F_0 $\cos^2$?
Angle of E-field to y –axis is now ? ?and angle to x–axis is ???/2 –??).
Passage though final polarizer leads to further reduction by a further factor $\cos^2$??/2 –??) = $\sin^2$? ?

✍ emergent flux is ½F_0 $\cos^2$? ?$\sin^2$? = (1/8) F_0 sin 2? ?

?

Maximum flux of F_0/8 is achieved for ? ?? ??/4

(c) Suppose the third linear polarizer in part (b) is now replaced by a half-wave plate, with its retardation axis at angle ? to that of the first polarizer's transmission axis. (Radiation polarized along the "retardation axis" is retarded by one-half wavelength relative to radiation polarized along a perpendicular axis.) What is now the emergent flux after the light has passed through all three filters? What is the maximum possible emergent flux, and for what ? is it obtained?

Half-wave plate flips sign of E-field amplitude along retardation axis
✍ polarization vector reflected about retardation axis but no change in flux

After passing through half-wave plate, radiation still has flux $\frac{1}{2}F_0$ but polarization axis is rotated by 2? relative to y–axis
✍ angle to x-axis is ??/2 –??)

✍ Emergent flux = $\frac{1}{2}F_0 \cos^2$??/2 –??) = $\frac{1}{2}F_0 \sin^2$ (??)

Maximum flux of $\frac{1}{2}F_0$ is achieved for ????/4

B4. (15 points) A Young's double slits experiment – which uses monochromatic light – is carried out with two slits of unequal width: Slit 2 has 4 times the width of Slit 1 and therefore emits 4 times the total power. (Both slits have width << ?)

Show that the dark fringes are not completely dark but have one-ninth the light intensity of the bright fringes.

Slit 2 emits 4 times the power of Slit 1
✍ E-field amplitude is twice as large: $E_2 = 2E_1$

Electric field at screen = $E_1 \cos(? t + ?_1) + E_2 \cos(? t + ?_2)$
Maxima obtained for $?_2 – ??_1 = 2?n$ have $|E| = |E_1 + E_2| = 3E_1$
Minima obtained for $?_2 – ??_1 = 2??n –1)$ have $|E| = |E_1 – E_2| = E_1$
✍ non-zero brightness

Ratio of intensities, minimum/maximum $= (E_1/ 3E_1)^2 = 1/9$