Lecture Notes on Waves – Part 3

- Last updated 12/4/02
- Pages 1 – 46
- Lectures 27 – 35
- Crawford Chapters 6 – 9

L27 →

Wave modulation

- How do we use EM waves to transmit information?

- We typically use a high frequency “carrier wave”, and then modulate (alter) it’s amplitude or frequency

Amplitude modulation (AM) – varying amplitude represents signal

Transmitter emits: $D(t) = A(t) \cos(\omega t)$

Amplitude varies slowly compared to $\cos(\omega t)$

Carrier wave

Example: Transmitting a pure cosine wave

- Amplitude $A(t) = A_0 \cos(\omega_{mod} t)$
- Transmitted output (at z=0):

$$D(t) = A_0 \cos(\omega_{mod} t) \cos(\omega t)$$

$$= \tfrac{1}{2} A_0 \cos(\omega_1 t) + \tfrac{1}{2} A_0 \cos(\omega_2 t)$$

where $\omega_1 = \omega - \omega_{mod}$ and $\omega_2 = \omega + \omega_{mod}$

- Displacement at position z:

$$\Psi(z,t) = \tfrac{1}{2} A_0 \cos(\omega_1 t - k_1 z) + \tfrac{1}{2} A_0 \cos(\omega_2 t - k_2 z)$$

$$= A_0 \cos(\omega_{mod} t - k_{mod} z) \cos(\omega t - kz)$$

where $k_{mod} = \tfrac{1}{2}(k_2 - k_1)$ and $k = \tfrac{1}{2}(k_2 + k_1)$

NB: $\cos A + \cos B = 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B)$

Signal speed

- The modulation propagates at speed

$$v_{group} = \omega_{mod} / k_{mod} \qquad \text{(the “group velocity”)}$$
$$= ½(\omega_2 - \omega_1) / ½(k_2 - k_1)$$
$$= d\omega/dk$$

- Note: for a non-dispersive medium,

$$v_{group} = d\omega/dk \;=\; \omega/k \;= v_\phi$$

But for a dispersive medium, the group and phase velocities need not be equal

Waves in a plasma

- We noted previously that EM waves in a plasma show the following dispersion relation:

  $\omega^2 - \omega_p{}^2 = c^2k^2$

  This leads to a phase velocity that exceeds c:

  $v_\phi{}^2 = \omega^2/k^2 = c^2 + \omega_p{}^2/k^2 > c^2$

  What about the group velocity?

  $v_g = d\omega/dk = c^2/v_\phi < c$

  AS REQUIRED BY RELATIVITY

Waves in 2-D and 3-D

- In 1-D, the wave equation (Klein-Gordon or classical) involves the term $\partial^2\Psi(z,t)/\partial z^2$

  Solution is $\Psi = A \cos(\omega t - kz + \phi)$,
  for which $\partial^2\Psi/\partial z^2 = -k^2\Psi$

- In 3-D, this term is replaced by
  $\nabla^2\Psi(x,y,z,t) \equiv \partial^2\Psi/\partial x^2 + \partial^2\Psi/\partial y^2 + \partial^2\Psi/\partial z^2$

  Solution is $A \cos(\omega t - [k_x x + k_y y + k_z z] + \phi)$
  for which $\nabla^2\Psi = -(k_x^2 + k_y^2 + k_z^2)\ \Psi$

Why the "Laplacian", $\nabla^2\Psi$?

- In fact, we have already considered 3-D waves (sound, EM) – we just chose the z axis to lie along the direction of propagation
- Suppose we change the orientation of our axes by the transformation (x,y,z) → (x',y',z')
- It can be shown that

  $\partial^2\Psi/\partial z^2 = \partial^2\Psi/\partial x'^2 + \partial^2\Psi/\partial y'^2 + \partial^2\Psi/\partial z'^2 = \nabla'^2\Psi$

  for any rotation of the axes

→ replace $\partial^2\Psi/\partial z^2$ with $\nabla^2\Psi$ to generalize wave equation

Quick review of vector operators

- Key vector operator: $\underline{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$

- Operates on a scalar function, $S(x,y,z)$
  - Direction of $\underline{\nabla}S$ (a.k.a."grad S") is perpendicular to the contours of constant S (with $\underline{\nabla}S$ pointing toward regions of higher S)
  - Magnitude of $\underline{\nabla}S$ is $\delta S/\delta l$ for motions perpendicular to the contours of constant S

What makes this a vector

- Key property of vectors: invariance of **scalar product** (“dot product”) under some transformation
  - For 4-vectors, that transformation is a change of IRF
  - For 3-vectors, that transformation is a rotation

- $\underline{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ has this property:
  - Suppose we move from location $\underline{r}$ to location $\underline{r} + d\underline{r}$
  - Regardless of how our axes are oriented, the resultant change in S is

    $dS = dx\,(\partial S/\partial x) + dy\,(\partial S/\partial y) + dz\,(\partial S/\partial x) = d\underline{r}.\underline{\nabla}S$

Scalar product is invariant

L28 →

Divergence

- The fact that $\underline{\nabla}$ is a vector implies that the “divergence” is invariant
  - Consider some vector “field” (i.e. a vector which is a function of r), $\underline{V}(\underline{r})$
  - Define the divergence of V as

$$\mathrm{div}\, V = \underline{\nabla}.\underline{V} = \partial V_x/\partial x + \partial V_y/\partial y + \partial V_z/\partial z$$

Being the scalar product of two vectors, this must be rotationally invariant

Laplacian

- Finally note that the Laplacian is simply the divergence of grad S:

  $$\nabla^2 S = \partial^2 S/\partial x^2 + \partial^2 S/\partial y^2 {}^{+} \partial^2 S/\partial z^2$$

  $$= \mathrm{div}\ (\mathrm{grad}\ S) = \underline{\nabla}.(\underline{\nabla} S)$$

  Being a scalar product, it must be invariant under rotations

The wave vector

- Classical 3-D wave equation: $\partial^2\Psi/\partial t^2 = v_\phi^2 \nabla^2\Psi$
- Its solution is

$$\begin{aligned}\Psi &= A\cos(\omega t - [k_x x + k_y y + k_z z] + \phi) \\ &= A\cos(\omega t - \underline{k}.\underline{r} + \phi)\end{aligned}$$

where $\underline{k} = (k_x, k_y, k_z)$ is the wave vector

Note that $\nabla^2\Psi = -(k_x^2 + k_y^2 + k_z^2)\,\Psi = -k^2\,\Psi$
where $k = |\underline{k}|$

As in the 1-D case, $v_\phi = \omega/k$

Plane waves

- General solution involves an infinite sum of waves of different ω and $\underline{k}$

- If all the wave vectors involved are coaligned, we have a plane wave, since for any term

  $\underline{\nabla}\,\Psi = -\,\underline{k}\,A\sin(\omega t - \underline{k}.\underline{r} + \phi)$

  ➔ planes perpendicular to $\underline{k}$ are loci of constant Ψ

Standing waves

- The 3-D wave equation has normal modes of the form: $\Psi = f\,(x,y,z)\cos(\omega t + \phi_t)$

  where $\nabla^2 f = -k^2 f = \omega^2 f / v_\phi^{\,2}$

  Solution: $f = \sin(k_x x + \phi_x)\sin(k_y y + \phi_y)\sin(k_z z + \phi_z)$ (separable solution)

  General solution: can have sum of terms with different (k_x, k_y, k_z) provided $(k_x^2 + k_y^2 + k_z^2)$ is fixed

  In HW10, you will find the normal modes for a 2-D system

L29 →

Mixed travelling and standing waves

- Consider the propagation of EM waves through a rectangular wave guide. Suppose the waves are polarized in the x-direction and propagate in the z-direction. The wave guide has width b in the y-direction (and a in the x-direction)

- E_X must vanish on conductive walls at $y = 0$, $y = b$

➔ $E_X = \sin(k_y y)\cos(k_z z - \omega t + \phi)$

with $k_y = m\pi/b$ and $\omega^2 = c^2 (k_y^2 + k_z^2)$

Low frequency cutoff for waveguide

- The minimum frequency is obtained for the lowest mode: $k_y = \pi/b$

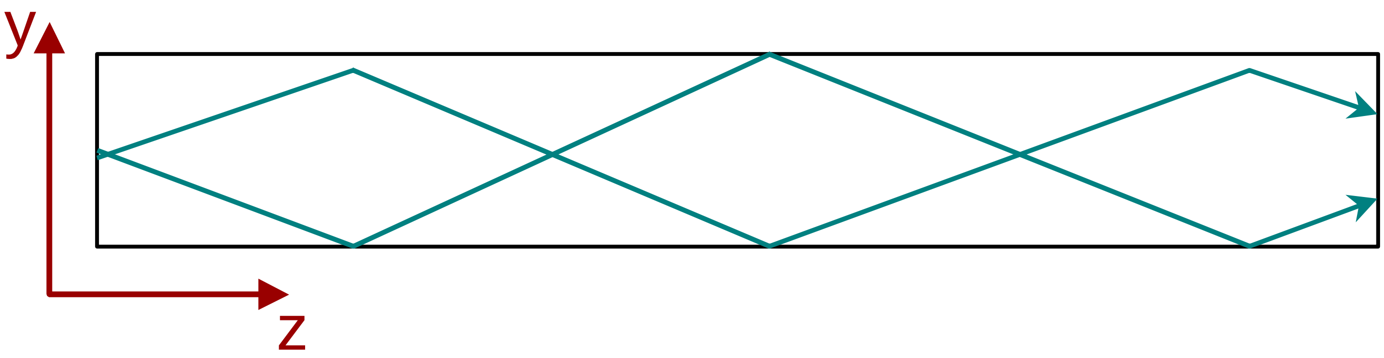
- Dispersion relation: $\omega^2 = \omega_0^2 + c^2 k_z^2$

  where $\omega_0^2 = c^2\pi^2/b^2$ is the low frequency cut-off frequency

- Phase velocity = $\omega/k_z = c/\sqrt{(1 - \omega_0^2/\omega^2)}$
- Group velocity = $d\omega/dk_z = c\sqrt{(1 - \omega_0^2/\omega^2)}$

Physical interpretation

- Criss-cross travelling waves



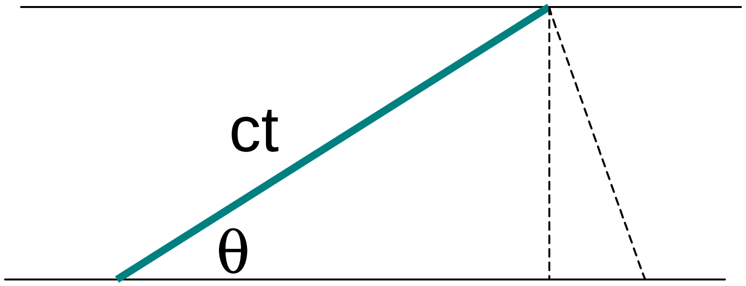
Rays inclined at angle $\cos\theta = k_z/k$ to z-axis,

where $\cos\theta = \sqrt{(1 - \omega_0^2/\omega^2)}$

L30 →

Phase and group velocity

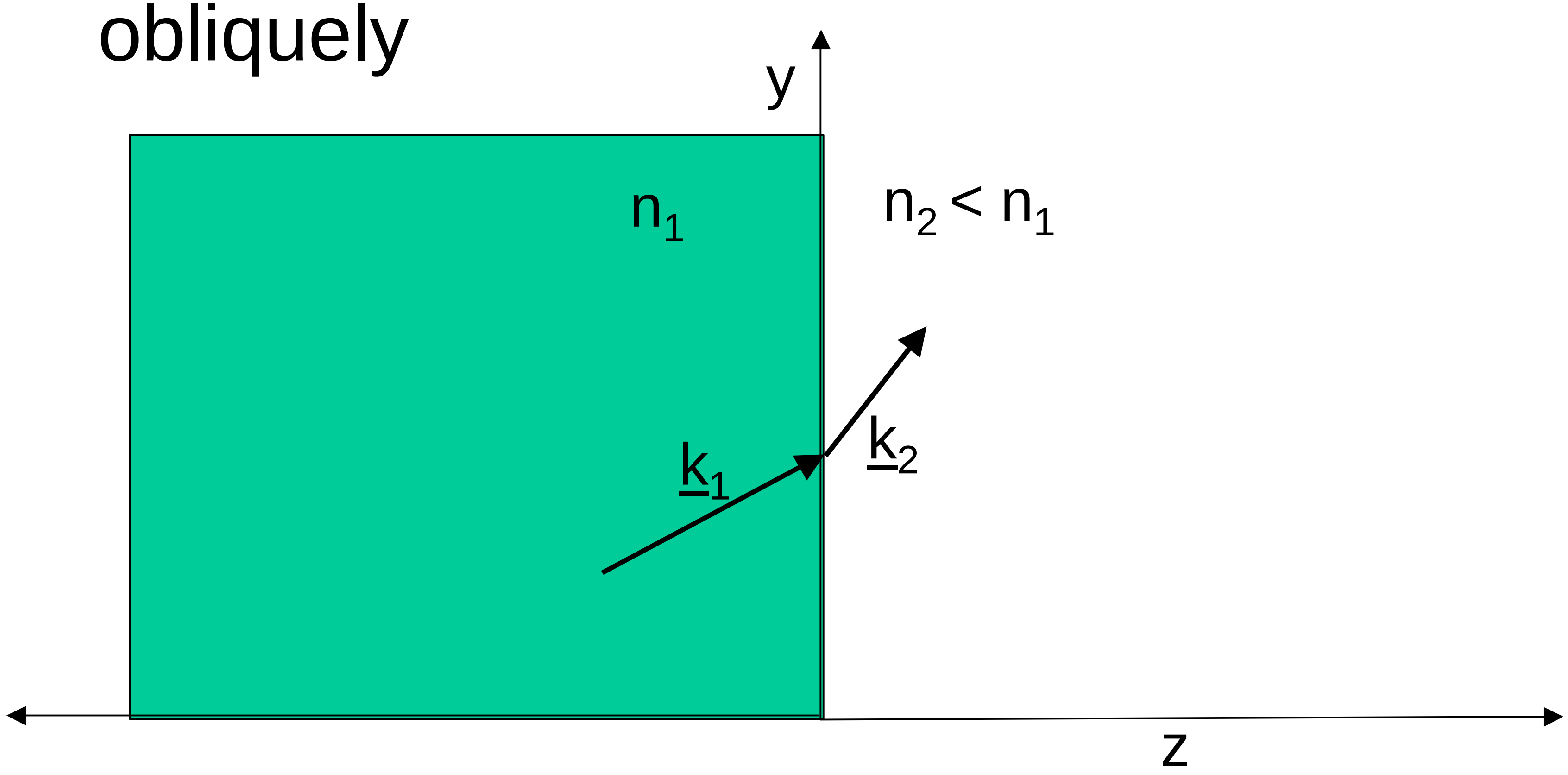
- Consider one section of one ray. Suppose it takes time t for signal to get from A to B



- z-distance travelled by signal in a given time is decreased by cos θ factor ➔ $v_g = c \cos\theta$
- z-distance between successive peaks or troughs is **increased** by sec θ factor ➔ $v_\phi = c / \cos\theta$

Refraction

- Consider a ray which hits an interface obliquely



Snell's Law

- **Boundary conditions**

(1) $\omega_1 = \omega_2 \Rightarrow k_1 c/n_1 = k_2 c/n_2$

(2) $k_{1y} = k_{2y} \Rightarrow k_1 \sin\theta_1 = k_2 \sin\theta_2$

- Substitute BC (1) into BC (2)

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 = (k_1 n_2/n_1) \sin\theta_2$$

$$\Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Total internal reflection

- Dispersion relation for medium 2

$$n_2^2\, \omega^2 / c^2 = k_2^2$$

$$\Rightarrow \quad \omega^2 = (c/n_2)^2 (k_{2z}^2 + k_{2y}^2)$$

$$= (c/n_2)^2 (k_{2z}^2 + k_{1y}^2)$$

$$= (c/n_2)^2 (k_{2z}^2 + k_1^2 \sin^2 \theta_1)$$

$$\omega^2 = (c/n_2)^2 k_{2z}^2 + \omega^2 (n_1/n_2)^2 \sin^2 \theta_1$$

$$\Rightarrow (c/n_2)^2 k_{2z}^2 = \omega^2 (1 - (n_1/n_2)^2 \sin^2 \theta_1)$$

If $\sin \theta_1 > n_2/n_1$, then k_{2z} is imaginary

$\Rightarrow$ exponential decay in z-direction

Solution for sin $\theta_1 > n_2/n_1$

- Solution for wave in medium (2):

$$\Psi = Z \exp(i\omega t - i\underline{k}.\underline{r})$$

$$= Z \cos(\omega t - k_y y + \phi) \exp(-\kappa z)$$

where $\kappa = ik_{2z} = \omega (n_2/c) [(n_1/n_2)^2 \sin^2 \theta_1 - 1]^{1/2}$

Amplitude drops by a factor e every time z increases by an amount

$$1/\kappa = (\lambda_2/2\pi) [(n_1/n_2)^2 \sin^2 \theta_1 - 1]^{-1/2}$$

Polarization

- Wave vector describes the propagation direction of a travelling wave
- In 2 or 3-D, the displacement itself has a direction
  - Longitudinal waves: $\underline{\Psi}$ is parallel to $\underline{k}$
  - Transverse waves: $\underline{\Psi}$ is perpendicular to $\underline{k}$

- In 3-D, transverse waves can have two orthogonal polarizations.
- If we define z as the propagation direction, $\underline{\Psi}$ can be directed along x or y (or anywhere in between)

Polarized waves

- In general, the polarization direction can vary in a random manner and there is no net average polarization → often true of EM radiation

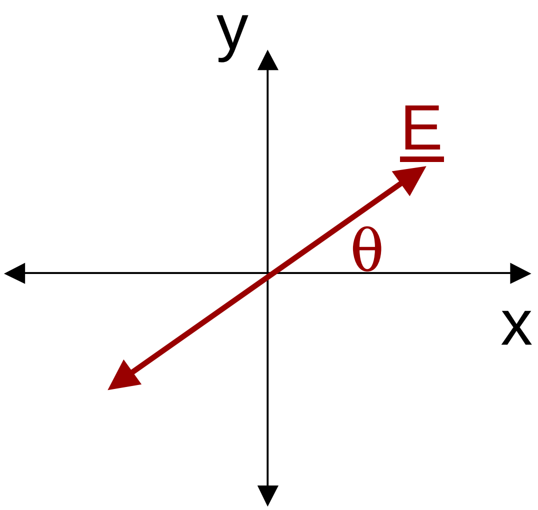
- In some cases, however, the polarization is constant. In this case, the radiation is said to be 100% polarized.

  (In the intermediate case, the polarization varies but still has a non-zero average.)

Linear polarization

- Electric field has a fixed orientation

$$\underline{E} = \underline{E}_0 \exp(i\omega t - ikz)$$
$$= (E_{0x}\,\hat{\underline{x}} + E_{0y}\,\hat{\underline{y}}) \exp(i\omega t - ikz)$$



$$E_{0x} = E_0 \cos\theta$$
$$E_{0y} = E_0 \sin\theta$$

$$E_{0y}/E_{0x} = \tan\theta$$
$$|E_{0x}|^2 + |E_{0y}|^2 = E_0^2$$

A linear polarizer

- A skein of wires acts as a polarizer for EM radiation with λ > wire spacing
- Orient wires along y-axis ➔ zero impedance for E_y ➔ E_y is reflected and only E_x is transmitted
- Transmitted radiation has

$$\begin{pmatrix} E_{0x}^{T} \\ E_{0y}^{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$

- Fraction of power transmitted = $(E_0^T/E_0)^2 = \cos^2\theta$
  - For unpolarized incident radiation, $\langle\cos^2\theta\rangle = ½$

Other polarizations

- Thus far, we have assumed that E_x and E_y vary in phase. But we could also have the superposition of two waves that are out of phase:

- Suppose these waves have equal amplitude and are $\pi/2$ out of phase

$$\underline{E} = A\,\hat{\underline{x}}\cos(\omega t - kz) + A\,\hat{\underline{y}}\sin(\omega t - kz)$$

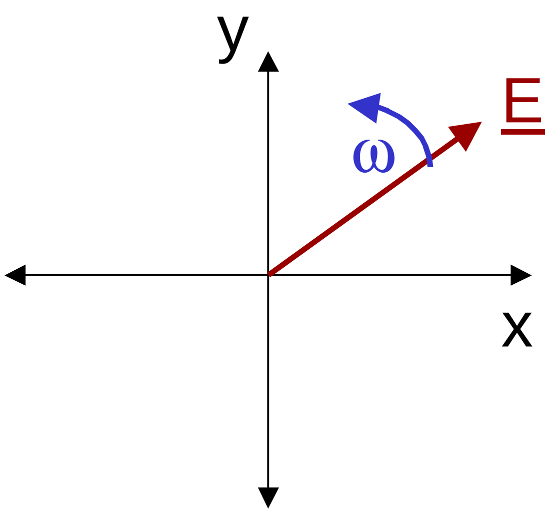
$$= (E_{0x}\,\hat{\underline{x}} + E_{0y}\,\hat{\underline{y}})\exp(i\omega t - ikz)$$

with $E_{0x} = A$ and $E_{0y} = -iA$

Circular polarization

- This is called circular polarization

$$\underline{E} = \underline{E}_0 \exp(i\omega t - ikz) = (E_{0x}\,\underline{\hat{x}} + E_{0y}\,\underline{\hat{y}}) \exp(i\omega t - ikz)$$



$$E_x = E_{0x} \cos(\omega t - kz)$$
$$E_y = E_{0x} \sin(\omega t - kz)$$

$$E_{0y}/E_{0x} = -i \text{ (or } + i)$$

$$\underline{E}_0 = E_0 \begin{pmatrix} +1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$$

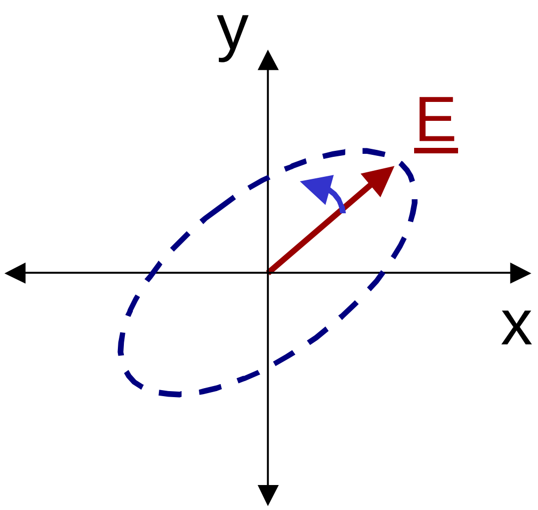
$\underline{E}$ executes circular motion

Circular polarization from linear

- Suppose we pass linearly polarized light with $E_{0x} = E_{0y}$ ($\underline{E}$ at 45 degree angle) through an anisotropic material possessing different v_ϕ for x and y polarizations.
  - In a "quarter-wave plate", we arrange for y polarization to be retarded by ¼λ
- Initial $\underline{E}_0 = E_0 \begin{pmatrix} +1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix}$ (linearly polarized)
- Final $\underline{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} +1/\sqrt{2} \\ +1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} +1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$ (circular)

Elliptical polarization

- The most general case of 100% polarized radiation has ***different*** amplitudes for the x and y components and ***arbitrary*** phase shifts (not just 0 or $\pi/2$)



Ratio of E_{0y} to E_{0x} is COMPLEX

$\underline{E}$ moves along an ellipse

Generation of linearly-polarized radiation

- Key mechanism: charges oscillating along the y-axis (for example) create y-polarized radiation

- Dipole antenna: moves current up and down a given axis ➔ linearly polarized radiation

- Scattering/reflection: see net polarization when we view scattering electrons along a ray inclined to incident ray

  ➔ usefulness of polarizing sunglasses

Interference

- Suppose we have two EM waves with an identical frequency, polarization and direction

$E_1 \cos(kz - \omega t)$

$E_2 \cos(kz - \omega t + \phi)$

Total E-field is

$$E = E_1 \cos(kz - \omega t) + E_2 \cos(kz - \omega t + \phi)$$

Total flux, $< S > \; = \; c\varepsilon_0 < E^2 >$

$$= c\varepsilon_0 \left(\tfrac{1}{2} E_1^2 + \tfrac{1}{2} E_2^2 + E_1 E_2 \cos\phi\right)$$

Constructive and destructive interference

- Case 1: $\phi = 0$ ➔ waves perfectly in phase

  ➔ $\langle S \rangle = \frac{1}{2} c \varepsilon_0 (E_1 + E_2)^2$

  $= \langle S_1 \rangle + \langle S_2 \rangle + 2 (\langle S_1 \rangle \langle S_2 \rangle)^{1/2}$

  "Constructive interference"

- Case 2: $\phi = \pi$ ➔ waves out of phase

  ➔ $\langle S \rangle = \frac{1}{2} c \varepsilon_0 (E_1 - E_2)^2$

  $= \langle S_1 \rangle + \langle S_2 \rangle - 2 (\langle S_1 \rangle \langle S_2 \rangle)^{1/2}$

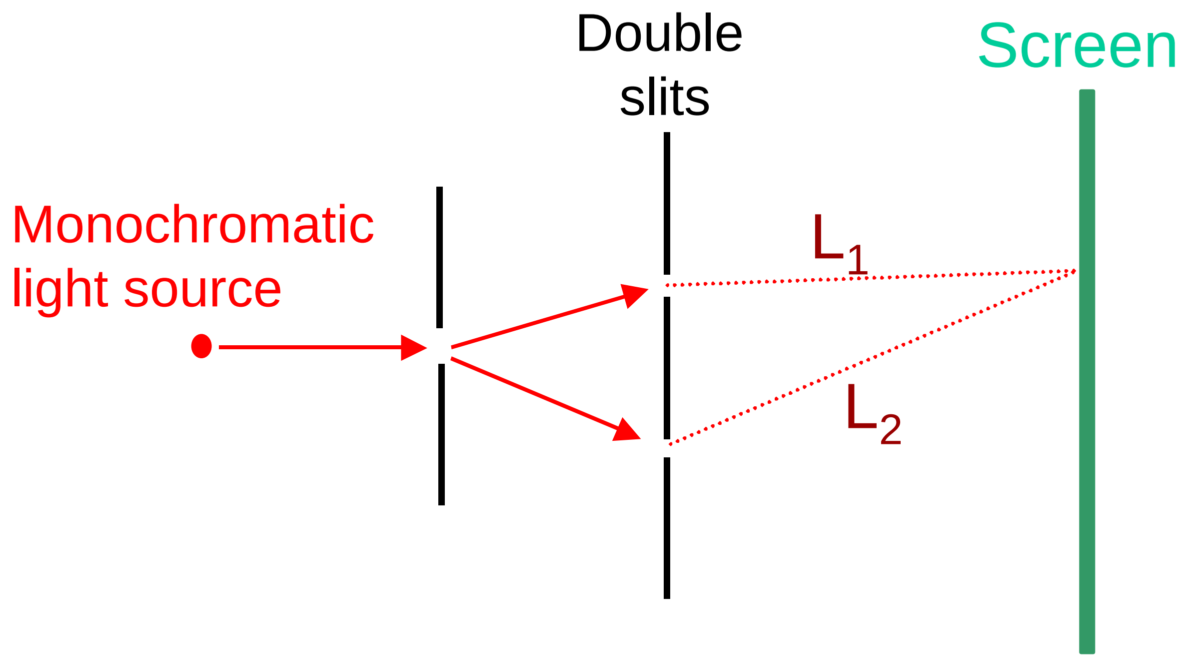
  "Destructive interference" ($\langle S \rangle$ drops to 0 if $E_1 = E_2$)

- Case 3: phase difference ϕ varies randomly with time ➔ $\langle E_1 E_2 \rangle = 0$

  ➔ $\langle S \rangle = \frac{1}{2} c \varepsilon_0 (E_1^2 + E_2^2) = \langle S_1 \rangle + \langle S_2 \rangle$

  "Incoherent radiation"

Young's slits experiment



Constructive interference if

$L_2 - L_1 = \Delta L = n\lambda$ (n is an integer)

Destructive interference if

$\Delta L = (n + ½)\,\lambda$ (n is an integer)

Comments on double slit experiment

1) If $L_1 \sim L_2$, then $E_1 = E_2$

minimum flux = 0

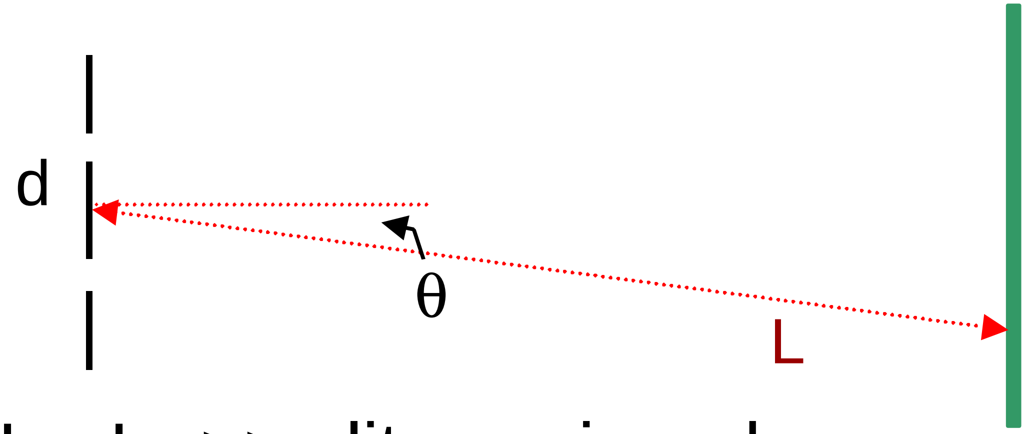
maximum flux = 4 $\langle S_1 \rangle$

average flux = 2 $\langle S_1 \rangle$ (energy conserved)

2) Original light source doesn't have to be coherent (phase may vary with time).

Phase difference between rays remains constant even if absolute phases vary.

Distant screen case



If L_1, L_2 >> slit spacing, d

$L_1 = L + ½\, d \sin\theta$ and $L_2 = L - ½\, d \sin\theta$

$|\Delta L| = d \sin\theta \Rightarrow \phi = (2\pi d/\lambda) \sin\theta$

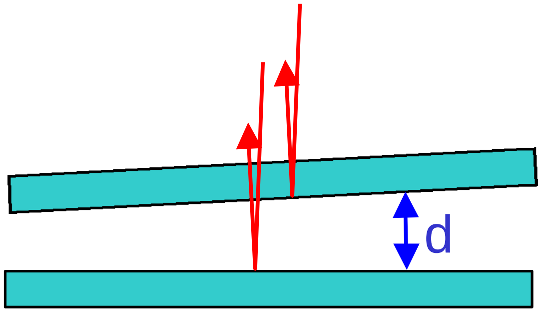
$< S > = c\varepsilon_0\, (½\, E_1^2 + ½\, E_2^2 + E_1\, E_2 \cos\phi\,)$

$= c\varepsilon_0\, E_1^2\, (1 + \cos\,([2\pi d/\lambda] \sin\theta))$

$= 4 < S_1 > \cos^2 (\pi d \sin\theta\, /\lambda)$

Interference between reflections

“Fringes” from interference between multiple reflections



Difference in path length = 2d

Difference in phase, $\phi = 4\pi d/\lambda + \pi$

Note: additional phase change π occurs when wave reflects off front surface of glass

Constructive interference if $d = (n + ½)\ \lambda/2$

Destructive interference if $d = n\lambda/2$

Other examples

Newton’s rings: interference rings with spherical and flat surfaces in contact

Thin-film interference: reflection of front and rear surfaces of a single thin film (e.g. a soap bubble, anti-reflection coatings)

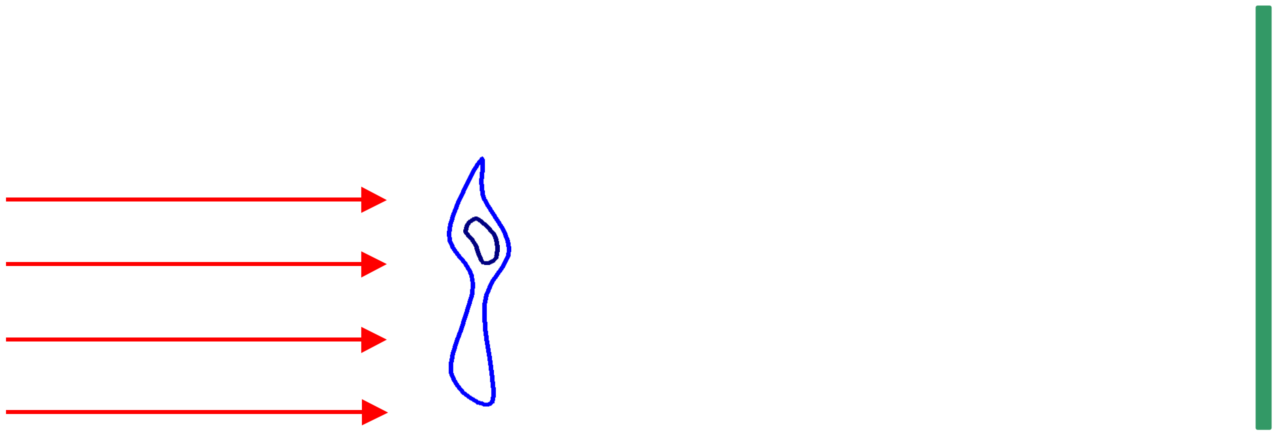
Fabry-Perot interferometer:

Multiple reflection between partially silvered mirrors. Fringe maxima become very well defined. Use as spectrometer by varying plate separation.

Diffraction

The double slit experiment is one example of the effect of placing an obstacle in a light path.

In general the obstacle could be of any shape:



For simplicity, we consider the case where the incident wave is a plane wave (source at infinity) and the screen is far away

Diffraction grating

Let’s extend the case of the double slit to that of a diffraction grating

Suppose we now have N equally-spaced slits with distance d between adjacent slits.

At a distant screen, the nth slit gives rise to an electric field $E = E_n \cos(\omega t + \phi_n)$,

where $\phi_n = (2\pi d/\lambda)(n-1) \sin\theta = 2(n-1)\beta$

and $\beta = (\pi d/\lambda) \sin\theta$

Diffraction grating equation

- If the screen is far from the slits, then E_n is the same for every slit and the total electric field is

$$E = E_1 \sum \cos(\omega t + 2[n-1]\beta)$$

- It can be shown that this sum is equal to

$$E = E_1 [\sin(N\beta)/\sin(\beta)] \cos(\omega t + [N+1]\beta)$$

Note that in the limit $\theta \rightarrow 0$ we have $\beta \rightarrow 0$ and

$E \rightarrow E_1 N \cos(\omega t)$ (as expected since all slits are in phase)

- Flux = $\langle S \rangle = \frac{1}{2} c\, \varepsilon_0 E^2$

$$= \tfrac{1}{2} c\, \varepsilon_0 E_1^2 [\sin(N\beta) / \sin(\beta)]^2$$

Derivation using complex numbers

$$E = E_1 \sum \cos(\omega t + 2[n-1]\beta)$$
$$= E_1 \mathrm{Re}\{\sum \exp[i(\omega t - 2\beta + 2n\beta)]\}$$
$$= E_1 \mathrm{Re}\{\exp[i(\omega t - 2\beta)] \sum (e^{i2\beta})^n\}$$

Note that the sum is a geometric progression, and use the result $\sum x^n = x(1 - x^N)/(1 - x)$

Hence $\sum (e^{i2\beta})^n = e^{i2\beta}(1 - e^{i2\beta N})/(1 - e^{i2\beta})$

$$= e^{i2\beta} e^{i\beta N}(e^{-i\beta N} - e^{i\beta N})/[e^{i\beta}(e^{-i\beta} - e^{i\beta})]$$
$$= e^{i\beta(N+1)} \sin(N\beta)/\sin(\beta)$$

and thus $E = E_1 \mathrm{Re}\{e^{i[\omega t + \beta(N+1)]} \sin(N\beta)/\sin(\beta)\}$

$$= E_1 \cos(\omega t + \beta(N+1)) \sin(N\beta)/\sin(\beta)$$

Notes on the diffraction grating

1) Local maxima with $F = \frac{1}{2} c\, \varepsilon_0 N E_1^2$
occur whenever $\beta = \pi n$, i.e. whenever $\sin \theta = n\lambda/d$

2) Position of maxima is determined by slit spacing and is independent of the number of slits

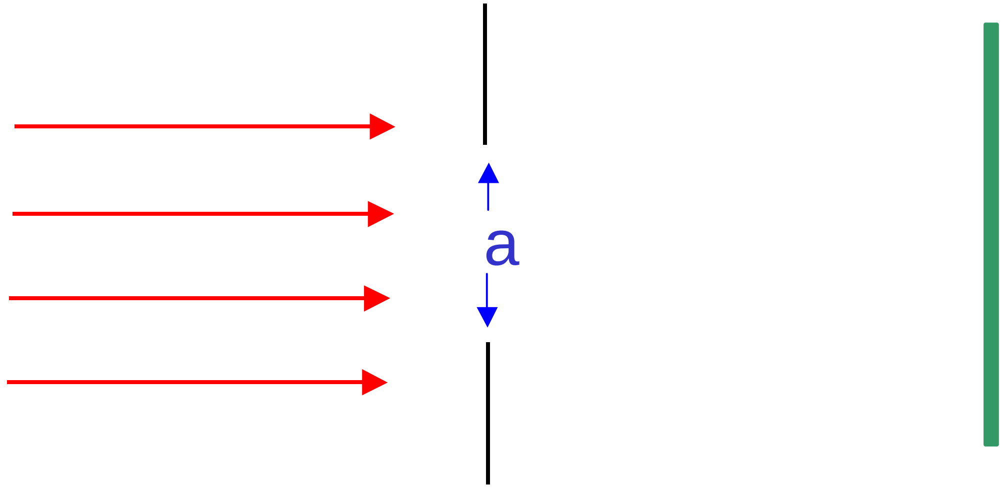
3) Amplitude of the maxima is proportional to N^2.

4) Width of the maxima is proportional to 1/N. Sharp maxima obtained for large N.

5) Mean flux is proportional to N, as expected from energy conservation.

Diffraction by a rectangular slit

Consider now a single slit of width,a



We can consider this to be a diffraction grating with a very large number of slits, N, of width d = a/N, corresponding to $\beta = \pi a \sin\theta / N\lambda$

Equation for diffraction by a slit

Diffraction grating equation

➔ $F = \frac{1}{2} c\varepsilon_0 E_1^2 [\sin(N\beta) / \sin(\beta)]^2$

$= \frac{1}{2} c\varepsilon_0 E_1^2 [\sin(N\beta) / \beta]^2$

(in the limit of large N, sin $\beta \rightarrow \beta$)

$= \frac{1}{2} c\, \varepsilon_0 E_1^2 N^2 \sin^2 a / a^2$

$= F(0) \sin^2 a / a^2$

where $a = N\beta = \pi a \sin\theta / \lambda$ and F(0) is the maximum flux (which occurs at $a = 0$).

Local minima occur at $a = n\pi$ $(|n| > 0)$ ➔ $\sin\theta = n\lambda/a$

The narrower the slit, the broader the diffraction pattern.

Diffraction through a circular aperture

Many optical instruments employ a circular aperture (of diameter D)

A detailed analysis of the interference between different parts of the aperture was worked out by George Airy and resulting pattern is called the Airy disk.

First minimum is at $\theta = 1.22\ \lambda/D$ rad (for $\lambda/D << 1$) and contains 85% of the light.

Diffraction limits the angular resolution achievable using an optical instrument.