Problem Set 7

Solution of Partial Differential Equation with I.C and B.C

For a VonNeumann analysis, the given flux equation \( \frac{\partial \rho}{\partial t} = -a \frac{\partial}{\partial x} (\rho \ln(\rho/\rho^*)) \)
is simplified as
\[ \frac{\partial \rho}{\partial t} = -a(\ln(\rho/\rho^*)-1) \frac{\partial \rho}{\partial x} \]

For space forward differencing,
\[ \frac{\partial \rho}{\partial t} = \frac{\rho^n_{j+1} - \rho^n_{j-1}}{2 \Delta x} \]

by substituting the eigen expansion \( \rho^n_j = \xi^n e^{ikx} \) into the differencing expression, we see that the right hand side of the equation has terms, which when expanded, the amplitude of the modulus factor will increase exponentially, making this difference technique unstable.

The result of the upwind differencing technique applied to the given problem is shown for \( \Delta t=0.05 \) and \( \Delta x=2.5 \) as a density plot. The colour scheme indicating the density at each \( t \) and \( x \) is also shown. The variation becomes much smoother for \( \Delta t=0.01 \) and \( \Delta x=0.5 \). Even at spatial steps \( \Delta x=2.5 \), one can still capture the physics of the problem.

The Courant condition \( |v| \frac{\Delta t}{\Delta x} \) is used for setting up the spatial and time steps.

![Figure 1(a): \( \rho(t,x) \) in part (b) \( \Delta t=0.05 \) and \( \Delta x=2.5 \), 0\( \leq x \leq 50 \) and 0\( \leq t \leq 10 \)](image)
Figure 1(a): \( \rho(t,x) \) in part (b) for \( \Delta t=0.01 \) and \( \Delta x=0.5 \), \( 0 \leq x \leq 50 \) and \( 0 \leq t \leq 10 \)

By comparing the result in for the two given boundary conditions, we see that the initial sinusoidal density falls much faster in the part(d) case than in part(b). In fact, in figure 1, where \( 15 \leq \rho(t,x) \leq 35 \), the sinusoidal variation of the boundary condition persists throughout most of the spatial range.

Figure 2(a): \( \rho(t,x) \) in part (d) using \( \Delta t=0.01 \) and \( \Delta x=0.5 \), \( 0 \leq x \leq 50 \) and \( 0 \leq t \leq 10 \)

In part (d) on the other hand, when \( 20 \leq \rho(t,x) \leq 60 \) the sinusoidal boundary condition decays very rapidly. Based on these observations, the characteristic density compares with the maxima of \( F(\rho) \sim 52 \) car km\(^{-1}\).