1. **Rotation curve (1 point).** Figure 1 shows the rotation curve of a galaxy, with individual contributions from the halo, the disk and the bulge. Why do the curves B, D and H not add up to the total circular speed shown by the solid line?

2. **Black hole (4 points).** A spherically symmetric globular cluster with an isotropic velocity dispersion tensor has a density profile

\[ \rho(r) = \frac{\rho_0}{(1 + r^2/b^2)^{5/2}}, \]

constant mass-to-light ratio, total mass \(10^6 M_\odot\) and core radius \(b = 5\) pc. Calculate the central line-of-sight velocity dispersion in km/sec. Suppose now there is a central black hole with \(M_{BH} = 10^4 M_\odot\). The radius of influence \(r_{BH}\) of the black hole is defined as the distance at which the orbital velocity around the black hole equals the velocity dispersion of the stellar system. How big is the radius of influence, in pc? How much mass (in solar masses) is within the sphere of influence? What is the line-of-sight velocity dispersion in km/sec at \(R = 0.5r_{BH}\)? Neglect the effects of two-body relaxation.

3. **Gas disks (1 point).** A disk galaxy is observed to have an HI disk, which extends out to very large distances and has a constant rotational velocity as a function of distance of 200 km/sec. Assuming that at large distances from the center the dark matter halo dominates the enclosed mass, assuming that the halo is spherically symmetric, and assuming that the velocity dispersion of dark matter particles is isotropic, what is that velocity dispersion, in km/sec? What is the mass of the halo within 20 kpc, in solar masses?

4. **Stellar wind (3 points).** A massive star produces a spherically symmetric outflow of ionized gas observed via optically thin emission lines. The velocity of the outflow is \(v_0\) and the surface brightness of the emission is \(\propto R^{-\alpha}\) at \(R > R_0\), with \(\alpha > 2\). What is the observed line-of-sight velocity dispersion of the emission lines as a function of \(R\) for \(R > R_0\)? What is the functional form of the line-of-sight velocity profile?

5. **Bob (1 point).** Bob, a high-school student working on an astronomy project, takes the
map of the constellations printed in a popular astronomy book, looks up the stellar type of all the stars mentioned on it on Wikipedia and finds that there are as many red stars as there are blue stars. He knows that red stars have low mass and blue stars have high mass. He concludes that the rate at which stars form is independent of mass. List at least two problems with this argument.

6. A two-component system (3 points). A spherically symmetric galaxy consists of stars and dark matter, and both components are important for producing the potential. Assume for simplicity that each component has isotropic velocity dispersion. Suppose stars have mass density and velocity dispersion $\rho_1(r), \sigma_1(r)$ and dark matter particles have density and velocity dispersion $\rho_2(r), \sigma_2(r)$.

(a) Write down the Jeans and Poisson equations for this system.

(b) Suppose that the ratio of stellar density to dark matter density is constant and equal to 0.2 (i.e., $\rho_1(r)/\rho_2(r) = 0.2$). What, if anything, can you say about the ratio of the velocity dispersions $\sigma_1(r)/\sigma_2(r)$?

7. More gas disks (1 point). A perfectly round elliptical galaxy is observed to have a gas disk near the center, perhaps due to a recent acquisition of a gas-rich companion galaxy. How does the velocity dispersion within the gas disk compare with the velocity dispersion of the stars at the same point in the galaxy? Is it greater, smaller, the same, and why?

8. Magellanic Clouds (1 point). Are the Magellanic Clouds affected by dynamical friction due to our Galaxy? If so, what are the particles inducing the dynamical friction on these two satellites? In the case of Magellanic Clouds, how big are $v_M$ (the velocity of the body being slowed down) and $v_m$ (the velocity of particles that are responsible for the dynamical friction)?

9. Dispersion relation (2 points). The dispersion relation for some type of perturbation of the system reads $\omega^2 = c^2k^2 + bk + a$. Are these oscillations growing, decaying, or staying the same as a function of time? Is this system stable or unstable to this type of perturbation?

10. Free-fall (2 points). [BT 3.4] Prove that if a homogeneous sphere of a pressureless fluid with density $\rho$ is released from rest it will collapse to a point in free-fall time $t_{ff} = \frac{1}{4} \sqrt{\frac{3\pi}{2G\rho}}$.

11. Sheared-sheet approximation (4 points). Much of the dynamics of disks can be derived by examining a simple analog system called the “sheared sheet”. Suppose you have a differentially rotating disk (e.g., one with a declining $\Omega(R)$). When you move into a reference frame that is comoving with a piece of the disk, the parts of the disk closer to the center than you are moving faster and overtake you, while the parts of the disk farther from the center lag more and more behind. Thus, to the first order (neglecting the curvature of the orbits) an initially square piece of disk becomes sheared into a parallelogram shape in that frame, hence the name. This approximation is widely used for example in studies of accretion disks.

Consider the motion of a particle in the $x - y$ plane that is subject to the potential $\Phi(x, y) = -2\Omega Ax^2$ and the Coriolis force $-2\Omega e_x \times v$, where $v = v_x e_x + v_y e_y$, with $e$ being the unit vectors
defining the Cartesian system, and \( \Omega, A, \) and \( B \equiv A - \Omega \) are constants. We identify the origin of this coordinate system \((0,0,0)\) with the Sun and \( A, B \) with Oort’s constants.

(a) Prove that a possible trajectory is \( x = x_0, \ y = y_0 + v_{y0}(x_0)t \) and find the speed \( v_{y0}(x_0) \). What kind of galactic orbit does this represent?

(b) Show that any trajectory can be written in the form \( x = x_0 + X \sin \kappa(t - t_0), \ y = y_0 + v_{y0}(x_0)t + Y \cos \kappa(t - t_0), \) and evaluate \( \kappa \) and \( X/Y \) in terms of \( A \) and \( \Omega \). Discuss the analogy with epicyclic orbits.

(c) Find two exact integrals of motion (i.e., functions of \( x, y, v_x, v_y \)) that are analogs of the \( z \)-component of the angular momentum and the energy in the rotating frame.

(d) What is the ratio of the axes of the velocity ellipsoid for a uniform population of stars at a given position?

(e) Can this system be used to analyze the asymmetric drift? If so, how?

12. HI in the outer Galaxy (2 points). Assume that the Milky Way and its distribution of HI gas are perfectly axisymmetric, flat, and in circular rotation. Furthermore suppose that the rotation speed for \( R > R_0 \) is independent of radius and equal to \( V_0 \). Sensitive searches in the range \( 90^\circ < l < 270^\circ \) show that 21-cm emission from HI is observed at velocities relative to the Local Standard of Rest between 0 and \(-V_e \sin l \) but not outside this range, where \( V_e < V_0 \). What is the explanation for this finding, and what can you deduce about the distribution of HI?

13. X-ray atmosphere (3 points). A spherical galaxy is observed to be an extended source of X-rays with a surface brightness distribution given by

\[
\Sigma_X(R) = \frac{\Sigma_0}{1 + R/a},
\]

where \( a \) is the core radius and \( \Sigma_0 \) is the central surface brightness. The X-ray emission is assumed to be due to thermal bremsstrahlung of hot gas in hydrostatic equilibrium in the galaxy potential, and the gas itself contributes a negligible amount of mass. Aside from the overall normalization, the spectra of the X-ray emission seem to be independent of where in the galaxy they are extracted. Calculate the functional form of the galactic potential at large distances from the center, at \( r \gg a \).

14. Hyper-velocity stars (2 points). Dr. J. Anderson from STScI is trying to measure the proper motions of hyper-velocity stars to confirm their origin in the Galactic center. This involves accurately measuring the positions of the candidate stars relative to background galaxies. Assuming that the stars are indeed coming from the Galactic center, that the data span 7 years and that the ACS camera’s Wide-Field Channel is used, by how many pixels are the stars expected to move from the first exposure to the last exposure?

15. Rayleigh stability criterion (3 points). Consider gas with an adiabatic equation of state in an arbitrary axisymmetric potential \( \Phi(R, z) \). Gas is in equilibrium in purely azimuthal rotation in this potential, with a radial density profile \( \rho_0(R, z) \) and a pressure profile \( P_0(R, z) \).
Self-gravity of the gas can be neglected.

(a) What is the rotational velocity of the gas $v_0(R, z)$?

(b) Consider in-plane perturbations around this rotation, meaning that the $z$-component of the perturbed velocity is 0. We will only consider perturbations that have $m = 0$ (and thus all $\partial/\partial\phi$ derivatives are 0) and only perturbations with wavelengths much smaller than local $R$ (what does this mean?). Demonstrate that the rotation is stable to such perturbations if and only if the angular momentum per unit mass increases outwards.

Fig. 1.— Rotation curve of a galaxy