1. **Proper motion errors (2 points).** Suppose that \( N \gg 1 \) measurements of the position of a star relative to several quasars are made at equal intervals \( \Delta t \), all with the same uncertainties. The total timespan or the baseline of the observations is therefore \( T = N\Delta t \). Assuming that the trigonometric parallax is negligible, how does the accuracy of the resulting proper motion determination scale with the length of the baseline? In other words, if the accuracy of the proper motion is \( \epsilon \propto T^{-a} \), what is \( a \)?

2. **Sound speed in Earth’s atmosphere (2 points).** In class we solved for the sound waves in air assuming \( \Phi = 0 \). But in the Earth’s atmosphere this is not true: \( -\nabla \Phi = g \), with constant \( g \). Include this term in the derivation and explain what happens to the sound waves and why.

3. **Solar rotation speed (4 points).** (a) Harris’s catalog of Galactic globular clusters we used in a previous homework gives the Galactic latitude and longitude of each cluster (in Part I) as well as its radial velocity relative to the Local Standard of Rest (in Part III). Use these data to make a kinematic estimate of the rotation speed of the LSR, assuming that the cluster system itself does not rotate. Your result should include error bars. This is a kinematic estimate, not a dynamic estimate, i.e., you do not need to use Newton’s laws. Does your result agree with the recent estimates? If not, what do you think may be the problem with this method? (b) Using the data in Part II of Harris’s table, estimate the distance to M31, assuming that the luminosity function of globular clusters is the same in the two galaxies and that the mean apparent magnitude of the M31 globular clusters is \( \langle m_V \rangle = 17.1 \).

4. **Softened disk (3 points).** Problem 6.5 from Binney & Tremaine (old edition) or problem 6.3 from Binney & Tremaine (new edition).

5. **Ideal Magnetohydrodynamics (5+1+1 points).** The equations of non-relativistic ideal MHD can be found in many places (e.g., on Wikipedia). Unfortunately, many places give them in SI rather than Gaussian (cgs) units. For compatibility with astrophysics literature I highly recommend the latter, so you need to be careful where you look. (You will also discover to your great dismay that some sources redefine the magnetic field to get rid of the \( 4\pi \) terms, \( B_{\text{new}} = B_{\text{Gauss}}/\sqrt{4\pi} \), which is yet a third set of units, so watch out for this.) Ideal MHD describes the behavior of fluids in magnetic fields, with a great simplification that the fluid is assumed to be infinitely conductive. This is an excellent approximation for many astrophysical plasmas, for example quasar accretion disks. Every fluid element consists of electrons and positively charged nuclei, so its average charge is neutral, but if you apply an electric field to such fluid element, electrons and nuclei immediately start flowing in opposite directions with essentially no resistance and screen the field out.

Let’s just write out the equations of non-relativistic ideal MHD in Gaussian (cgs) units. We
start with the continuity equation, and it is the same as the one we discussed in class:

$$\frac{\partial \rho}{\partial t} + \nabla (\rho \mathbf{v}) = 0. \quad (1)$$

Euler equation acquires an Amper force term which should be familiar to you from basic E&M (just remember that it is per unit volume):

$$\rho \left( \frac{\partial}{\partial t} + (\mathbf{v} \nabla) \right) \mathbf{v} = \frac{\mathbf{j} \times \mathbf{B}}{c} - \nabla p - \rho \nabla \Phi. \quad (2)$$

Here $\mathbf{j}$ is the current density (the standard current flowing through the wires that you used in basic E&M is $I = \text{current density times the cross-section of the wire}$). We have neglected the electrostatic force here because we’ve assumed that the plasma or the fluid has 0 net charge density. In addition, to describe the E&M fields we have three of the four Maxwell’s equations which also should be familiar:

$$\nabla \mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad (4)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}. \quad (5)$$

It turns out that in the non-relativistic case the displacement current term can be neglected. If you don’t remember what this is, look it up and cross it off; we may discuss why this term is negligible at some later point, but for now let’s just get rid of it.

The last Maxwell’s equation connects the electric field with the charge density, which would then need its own equations which would be a big mess. Fortunately, under the assumptions of ideal MHD the electric field is screened because of the infinite conductivity in the frame co-moving with the fluid:

$$\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = 0. \quad (6)$$

(The field transforms from one frame to another can be found in many places, e.g., in Jackson’s Electrodynamics.) This equation replaces the 4th Maxwell’s equation. The final equation that wraps the system is the equation of state:

$$P = P(\rho). \quad (7)$$

These are the equations that are solved numerically when people study the behavior of non-relativistic plasmas in magnetic fields.

Our steady-state solution is homogeneous fluid $\rho_0$ at rest $\mathbf{v}_0 = 0$ with no gravity $\Phi = 0$ and uniform magnetic field $\mathbf{B}_0$ threading the fluid. $\mathbf{B}_0$ is directed for example along the $z$-axis (although you should try to keep the vector notation as long as possible). In addition we will consider the following simplifications:
• We will consider a fluid that is \textit{incompressible}: its density is always \( \rho_0 \) (so \( \rho_1 = 0 \) in perturbation analysis). This is for example a good approximation for liquid metals that are used in labs to study MHD experimentally: metals are highly conductive, nearly incompressible, and if the metal is liquid at room temperature it makes the experiment set up that much easier. For example, Princeton Plasma Physics Lab has an MHD turbulence experiment filled with liquid gallium.

• We will ignore the pressure terms in the Euler (or momentum) equation, so we will consider them subdominant to all other forces.

• We will drop the displacement current which is unimportant in non-relativistic motion.

• We will only consider the perturbations in which \( \mathbf{B}_1 \perp \mathbf{B}_0 \).

(a) First, determine to your satisfaction that the steady-state solution is in fact a solution to all the equations. Now let’s see if this solution is stable. Conduct the linear stability analysis of this solution to perturbations with \( \mathbf{B}_1 \perp \mathbf{B}_0 \). Derive the dispersion relation for these perturbations, find their phase and group velocity. Which way are they propagating? Which way are the fluid elements moving? Are these waves longitudinal or transverse? (I.e., are the fluid elements moving parallel or perpendicular to the direction of the wave propagation?) Are they growing or decaying? What are they called? (Hint: there are no curvy derivatives anywhere in this problem! If you need to use coordinates, use the Cartesian ones.)

(b) Calculate the group velocity of these perturbations for the HII gas in the disk of the Milky Way. Some information on the phases of the interstellar medium is summarized in the intro to the book by B.Draine “Physics of the interstellar and intergalactic medium” (although you still may need to look up \( B_0 \)). Compare this velocity to the sound speed in the same gas. Provide references for numerical values if necessary.

(c) Are ideal MHD equations appropriate for describing the structure and evolution of a protoplanetary disk? Why?