

Quantum Mechanics 171.605: Problem Set 3

Due: Friday, October 7, 2011 (in class)

P1. Consider a harmonic oscillator of mass m and angular frequency ω . At time $t = 0$, the state of this oscillator is given by:

$$|\psi(0)\rangle = \sum_n c_n |\varphi_n\rangle$$

where the states $|\varphi_n\rangle$ are stationary states with energies $(n + 1/2)\hbar\omega$.

a. What is the probability \mathcal{P} that a measurement of the oscillator's energy performed at an arbitrary time $t > 0$, will yield a result greater than $2\hbar\omega$? When $\mathcal{P} = 0$, what are the non-zero coefficients c_n ?

b. From now on, assume that only c_0 and c_1 are different from zero. Write the normalization condition for $|\psi(0)\rangle$ and the mean value $\langle H \rangle$ of the energy in terms of c_0 and c_1 . With the additional requirement $\langle H \rangle = \hbar\omega$, calculate $|c_0|^2$ and $|c_1|^2$.

c. As the normalized state vector $|\psi(0)\rangle$ is defined only to within a global phase factor, we fix this factor by choosing c_0 real and positive. We set: $c_1 = |c_1| e^{i\theta_1}$. We assume that $\langle H \rangle = \hbar\omega$ and that:

$$\langle X \rangle = \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}}$$

Calculate θ_1 .

d. With $|\psi(0)\rangle$ so determined, write $|\psi(t)\rangle$ for $t > 0$ and calculate the value of θ_1 at t . Deduce the mean value $\langle X \rangle(t)$ of the position at t .

P2. In a three-dimensional problem, consider a particle of mass m and of potential energy:

$$V(X, Y, Z) = \frac{m\omega^2}{2} \left[\left(1 + \frac{2\lambda}{3}\right) (X^2 + Y^2) + \left(1 - \frac{4\lambda}{3}\right) Z^2 \right]$$

where ω and λ are constants which satisfy:

$$\omega \geq 0 \quad , \quad 0 \leq \lambda < \frac{3}{4}$$

a. What are the eigenstates of the Hamiltonian and the corresponding energies?

b. Calculate and discuss, as functions of λ , the variation of the energy, the parity and the degree of degeneracy of the ground state and the first two excited states.

P3. The evolution operator $U(t, 0)$ of a one-dimensional harmonic oscillator is written:

$$U(t, 0) = e^{-iHt/\hbar}$$

with:

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$$

a. Consider the operators:

$$\tilde{a}(t) = U^\dagger(t, 0) a U(t, 0)$$

$$\tilde{a}^\dagger(t) = U^\dagger(t, 0) a^\dagger U(t, 0)$$

By calculating their action on the eigenkets $|\varphi_n\rangle$ of H , find the expression for $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ in terms of a and a^\dagger .

b. Calculate the operators $\tilde{X}(t)$ and $\tilde{P}(t)$ obtained from X and P by the unitary transformation:

$$\tilde{X}(t) = U^\dagger(t, 0) X U(t, 0)$$

$$\tilde{P}(t) = U^\dagger(t, 0) P U(t, 0)$$

How can the relations so obtained be interpreted?

c. Show that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right) |x\rangle$ is an eigenvector of P and specify its eigenvalue.

Similarly, establish that $U^\dagger\left(\frac{\pi}{2\omega}, 0\right) |p\rangle$ is an eigenvector of X .

d. At $t = 0$, the wave function of the oscillator is $\psi(x, 0)$. How can one obtain from $\psi(x, 0)$ the wave function of the oscillator at all subsequent times $t_q = q\pi/2\omega$ (where q is a positive integer)?

e. Choose for $\psi(x, 0)$ the wave function $\varphi_n(x)$ associated with a stationary state. From the preceding question derive the relation which must exist between $\varphi_n(x)$ and its Fourier transform $\bar{\varphi}_n(p)$.

f. Describe qualitatively the evolution of the wave function in the following cases:

(i) $\psi(x, 0) = e^{ikx}$ where k , real, is given.

(ii) $\psi(x, 0) = e^{-\rho x}$ where ρ is real and positive.

(iii)

$$\psi(x, 0) \begin{cases} = \frac{1}{\sqrt{a}} & \text{if } -\frac{a}{2} \leq x \leq \frac{a}{2} \\ = 0 & \text{everywhere else} \end{cases}$$

(iv) $\psi(x, 0) = e^{-\rho^2 x^2}$ where ρ is real.

P4. A one-dimensional harmonic oscillator is composed of a particle of mass m , charge q and potential energy $V(X) = \frac{1}{2} m\omega^2 X^2$. We assume in this exercise that the particle is placed in an electric field $\mathcal{E}(t)$ parallel to Ox and time-dependent, so that to $V(x)$ must be added the potential energy:

$$W(t) = -q\mathcal{E}(t)X$$

a. Write the Hamiltonian $H(t)$ of the particle in terms of the operators a and a^\dagger . Calculate the commutators of a and a^\dagger with $H(t)$.

b. Let $\alpha(t)$ be the number defined by:

$$\alpha(t) = \langle \psi(t) | a | \psi(t) \rangle$$

where $|\psi(t)\rangle$ is the normalized state vector of the particle under study. Show from the results of the preceding question that $\alpha(t)$ satisfies the differential equation:

$$\frac{d}{dt} \alpha(t) = -i\omega \alpha(t) + i\lambda(t)$$

where $\lambda(t)$ is defined by:

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}} \mathcal{E}(t)$$

Integrate this differential equation. At time t , what are the mean values of the position and momentum of the particle?

c. The ket $|\varphi(t)\rangle$ is defined by:

$$|\varphi(t)\rangle = [a - \alpha(t)] |\psi(t)\rangle$$

where $\alpha(t)$ has the value calculated in b. Using the results of questions a and b, show that the evolution of $|\varphi(t)\rangle$ is given by:

$$i\hbar \frac{d}{dt} |\varphi(t)\rangle = [\dot{H}(t) + \hbar\omega] |\varphi(t)\rangle$$

How does the norm of $|\varphi(t)\rangle$ vary with time?

d. Assuming that $|\psi(0)\rangle$ is an eigenvector of a with the eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of a , and calculate its eigenvalue.

Find at time t the mean value of the unperturbed Hamiltonian

$$H_0 = H(t) - W(t)$$

as a function of $\alpha(0)$. Give the root-mean-square deviations ΔX , ΔP and ΔH_0 ; how do they vary with time?

e. Assume that at $t = 0$, the oscillator is in the ground state $|\varphi_0\rangle$. The electric field acts between times 0 and T and then falls to zero. When $t > T$, what is the evolution of the mean values $\langle X \rangle(t)$ and $\langle P \rangle(t)$? Application: assume that between 0 and T , the field $\mathcal{E}(t)$ is given by $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega' t)$; discuss the phenomena observed (resonance) in terms of $\Delta\omega = \omega' - \omega$. If, at $t > T$, the energy is measured, what results can be found, and with what probabilities?