Quantum Mechanics 171.605: Problem Set 4

Due: Friday, October 21, 2011 (in conference)

P1. Consider a physical system whose state space, which is three-dimensional, is spanned by the orthonormal basis formed by the three kets $|u_1\rangle$, $|u_2\rangle$, $|u_3\rangle$. In this basis, the Hamiltonian operator H of the system and the two observables A and B are written:

$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ω_0, a and b are positive real constants.

The physical system at time $t = 0$ is in the state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

a. At time $t = 0$, the energy of the system is measured. What values can be found, and with what probabilities? Calculate, for the system in the state $|\psi(0)\rangle$, the mean value $\langle H \rangle$ and the root-mean-square deviation ΔH.

b. Instead of measuring H at time $t = 0$, one measures A; what results can be found, and with what probabilities? What is the state vector immediately after the measurement?

c. Calculate the state vector $|\psi(t)\rangle$ of the system at time t.

d. Calculate the mean values $\langle A \rangle(t)$ and $\langle B \rangle(t)$ of A and B at time t. What comments can be made?

e. What results are obtained if the observable A is measured at time t? Same question for the observable B. Interpret.

P2. Consider an arbitrary physical system. Denote its Hamiltonian by $H_0(t)$ and the corresponding evolution operator by $U_0(t, t')$:

$$\begin{cases} i\hbar \dfrac{\partial}{\partial t} U_0(t, t_0) = H_0(t)\, U_0(t, t_0) \\ U_0(t_0, t_0) = \mathbb{1} \end{cases}$$

Now assume that the system is perturbed in such a way that its Hamiltonian becomes:

$$H(t) = H_0(t) + W(t)$$

(Continued →)

The state vector of the system in the "interaction picture", $|\psi_I(t)\rangle$, is defined from the state vector $|\psi_S(t)\rangle$ in the Schrödinger picture by:

$$|\psi_I(t)\rangle = U_0^\dagger(t, t_0)\,|\psi_S(t)\rangle$$

a. Show that the evolution of $|\psi_I(t)\rangle$ is given by:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_I(t)\rangle = W_I(t)\,|\psi_I(t)\rangle$$

where $W_I(t)$ is the transform operator of $W(t)$ under the unitary transformation associated with $U_0^\dagger(t, t_0)$:

$$W_I(t) = U_0^\dagger(t, t_0)\,W(t)\,U_0(t, t_0)$$

Explain qualitatively why, when the perturbation $W(t)$ is much smaller than $H_0(t)$, the motion of the vector $|\psi_I(t)\rangle$ is much slower than that of $|\psi_S(t)\rangle$.

b. Show that the preceding differential equation is equivalent to the integral equation:

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t}\mathrm{d}t'\,W_I(t')\,|\psi_I(t')\rangle$$

where: $|\psi_I(t_0)\rangle = |\psi_S(t_0)\rangle$.

c. Solving this integral equation by iteration, show that the ket $|\psi_I(t)\rangle$ can be expanded in a power series in W of the form:

$$|\psi_I(t)\rangle = \left\{\mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t}\mathrm{d}t'\,W_I(t') + \frac{1}{(i\hbar)^2}\int_{t_0}^{t}\mathrm{d}t'\,W_I(t')\int_{t_0}^{t'}\mathrm{d}t''\,W_I(t'') + \ldots\right\}|\psi_I(t_0)\rangle$$