Quantum Mechanics 171.605: Problem Set 6

Due Friday, November 4, 2011 (in conference)

P1. Consider an arbitrary physical system whose four-dimensional state space is spanned by a basis of four eigenvectors $|j, m_z\rangle$ common to $\mathbf{J}^2$ and J_z ($j = 0$ or 1; $-j \leqslant m_z \leqslant +j$), of eigenvalues $j(j+1)\hbar^2$ and $m_z\hbar$, such that:

$$J_\pm |j, m_z\rangle = \hbar\sqrt{j(j+1) - m_z(m_z \pm 1)}\,|j, m_z \pm 1\rangle$$
$$J_+ |j, j\rangle = J_- |j, -j\rangle = 0$$

a Express in terms of the kets $|j, m_z\rangle$, the eigenstates common to $\mathbf{J}^2$ and J_x, to be denoted by $|j, m_x\rangle$

b Consider a system in the normalized state:

$$|\psi\rangle = \alpha\,|j = 1, m_z = 1\rangle + \beta\,|j = 1, m_z = 0\rangle + \gamma\,|j = 1, m_z = -1\rangle + \delta\,|j = 0, m_z = 0\rangle$$

(*i*) What is the probability of finding $2\hbar^2$ and $\hbar$ if $\mathbf{J}^2$ and J_x are measured simultaneously?

(*ii*) Calculate the mean value of J_z when the system is in the state $|\psi\rangle$, and the probabilities of the various possible results of a measurement bearing only on this observable

(*iii*) Same questions for the observable $\mathbf{J}^2$ and for J_x

(*iv*) J_z^2 is now measured; what are the possible results, their probabilities, and their mean value?

P2. Consider a system of angular momentum $l = 1$. A basis of its state space is formed by the three eigenvectors of L_z: $|+1\rangle$, $|0\rangle$, $|-1\rangle$, whose eigenvalues are, respectively, $+\hbar$, 0, and $-\hbar$, and which satisfy:

$$L_\pm |m\rangle = \hbar\sqrt{2}\,|m \pm 1\rangle$$
$$L_+ |1\rangle = L_- |-1\rangle = 0$$

This system, which possesses an electric quadrupole moment, is placed in an electric field gradient, so that its Hamiltonian can be written:

$$H = \frac{\omega_0}{\hbar}(L_u^2 - L_v^2)$$

where L_u and L_v are the components of $\mathbf{L}$ along the two directions Ou and Ov of the xOz plane which form angles of 45° with Ox and Oz; ω_0 is a real constant.

a Write the matrix which represents H in the $\{|+1\rangle, |0\rangle, |-1\rangle\}$ basis What are the stationary states of the system, and what are their energies? (These states are to be written $|E_1\rangle$, $|E_2\rangle$, $|E_3\rangle$, in order of decreasing energies)

b At time $t = 0$, the system is in the state:

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}[\,|+1\rangle - |-1\rangle\,]$$

What is the state vector $|\psi(t)\rangle$ at time t? At t, L_z is measured; what are the probabilities of the various possible results?

c Calculate the mean values $\langle L_x\rangle(t)$, $\langle L_y\rangle(t)$ and $\langle L_z\rangle(t)$ at t What is the motion performed by the vector $\langle \mathbf{L}\rangle$?

d At t, a measurement of L_z^2 is performed

(*i*) Do times exist when only one result is possible?

(*ii*) Assume that this measurement has yielded the result $\hbar^2$. What is the state of the system immediately after the measurement? Indicate, without calculation, its subsequent evolution

P3. Let **J** be the angular momentum operator of an arbitrary physical system whose state vector is $|\psi\rangle$

a Can states of the system be found for which the root-mean-square deviations ΔJ_x, ΔJ_y and ΔJ_z are simultaneously zero?

b Prove the relation:

$$\Delta J_x \ \Delta J_y \geqslant \frac{\hbar}{2} |\langle J_z \rangle|$$

and those obtained by cyclic permutation of x, y, z

Let $\langle \mathbf{J} \rangle$ be the mean value of the angular momentum of the system. The $Oxyz$ axes are assumed to be chosen in such a way that $\langle J_x \rangle = \langle J_y \rangle = 0$ Show that:

$$(\Delta J_x)^2 + (\Delta J_y)^2 \geqslant \hbar \, |\langle J_z \rangle|$$

c Show that the two inequalities proven in question *b*. both become equalities if and only if $J_+ |\psi\rangle = 0$ or $J_- |\psi\rangle = 0$

d The system under consideration is a spinless particle for which $\mathbf{J} = \mathbf{L} = \mathbf{R} \times \mathbf{P}$ Show that it is not possible to have both $\Delta L_x \ \Delta L_y = \frac{\hbar}{2} |\langle L_z \rangle|$ and $(\Delta L_x)^2 + (\Delta L_y)^2 = \hbar \, |\langle L_z \rangle|$ unless the wave function of the system is of the form:

$$\psi(r, \theta, \varphi) = F(r, \sin\theta \, e^{\pm i\varphi})$$