Relativity Restored: Dirac Anisotropy in QED$_3$

O. Vafek,[1] Z. Tešanović,[1] and M. Franz[2]
[1]*Department of Physics and Astronomy, Johns Hopkins University, Baltimore, Maryland 21218*
[2]*Department of Physics and Astronomy, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z*
(Received 11 March 2002; published 23 September 2002)

We show that, at long length scales and low energies and to leading order in $1/N$ expansion, the anisotropic QED in $2+1$ dimensions renormalizes to an isotropic limit. Consequently, the (Euclidean) relativistic invariance of the theory is spontaneously restored at the isotropic critical point.

DOI: 10.1103/PhysRevLett.89.157003 PACS numbers: 74.60.–w, 11.10.Kk, 11.15.Pg

Quantum electrodynamics in $(2+1)$ dimensions (QED$_3$) has recently emerged as a low-energy effective theory of a number of condensed matter systems [1–8]. Examples range from fluctuating d-wave superconductors in underdoped cuprates [4–7] to doped Mott insulators [3] to Heisenberg antiferromagnets and spin liquids [1,2,8]. While these multiple reincarnations of QED$_3$ differ mightily in their physical content, they all share certain important formal similarities. The low-energy behavior is controlled by an infrared fixed point where the gauge field acquires a universal dimensionless coupling constant $g \propto 1/N$, N being the number of Dirac fermion flavors [9,10]. At values of g larger than some critical value g_c ($N < N_c$) it is believed that the theory has an instability into a state with broken chiral symmetry, with fermions spontaneously acquiring a finite dynamical mass [9]. Among the formal aspects shared by the above theories surely one of the most ubiquitous is the spacetime anisotropy—such low-energy effective theories are obviously only pretending to be "relativistic." They hail from nonrelativistic quantum Hamiltonians and are not obliged to be invariant under Lorentz transformations. Consequently, they often contain more than one "speed of light" resulting in the above anisotropy. An important question is to what extent are the properties of these effective theories similar to the genuine, isotropic QED$_3$ and, in particular, what is the nature of the critical behavior and chiral symmetry breaking when such anisotropy is present.

In this Letter we address the problem of Dirac anisotropy in QED$_3$. Our point of departure is the assumption that the *symmetric* (massless or critical) phase of *isotropic* QED$_3$, obtained when the number of fermion flavors is larger than a critical value, $N > N_c$, is controlled by a stable nontrivial infrared critical point [9,10]. Based on this assumption, which is almost certainly correct for $N \gg N_c$, we derive the following results within a $1/N$ expansion: (i) When *anisotropy* is turned on at this interacting critical point we find it to be *marginally irrelevant* in a perturbative sense. This implies that the symmetric, critical phase of QED$_3$ remains unaffected by small Dirac anisotropy. And (ii), going beyond the perturbative regime, and by exploring the structure of renormalization group (RG) flows, we argue that any *finite* anisotropy is also irrelevant. Our results imply that the relativistic invariance of a QED$_3$-like effective theory is itself an *emergent* property: it is spontaneously dynamically restored at the critical point.

The anisotropic QED$_3$ can be defined as follows:

$$\mathcal{L} = \bar{\psi}^{(n)}[\gamma_\mu \sqrt{g^n}_{\mu\nu}(\partial_\nu + ia_\nu)]\psi^{(n)} + \frac{1}{2e^2}(\partial \times a)^2, \quad (1)$$

where $\psi^{(n)}$ is a Fermi field associated with a node n, γ_μ is a Dirac matrix, and a_μ is a massless U(1) gauge field related to fluctuations of unbound $2+1$ vortex loops [4–6]. We also introduced the diagonal "nodal" metric $g^n_{\mu\nu}$: $g^{(1)}_{00} = g^{(2)}_{00} = 1$, $g^{(1)}_{11} = g^{(2)}_{22} = v_F^2$, $g^{(1)}_{22} = g^{(2)}_{11} = v_\Delta^2$. Other forms can be reduced to this one by suitable rescalings of spacetime coordinates and fermion and gauge fields.

The Dirac anisotropy of $\mathcal{L}$ (1) is more sinister than its bosonic kin [11]. In the Higgs-Abelian gauge theory the anisotropy can be fully rescaled out of the matter part of the action leading to the new effective action with the anisotropy stored only in the gauge field Maxwellian action. Since matter cannot generate any anisotropic contribution to the gauge field and by virtue of the isotropic charge being a relevant operator, it is easy to show that anisotropy of the bosonic theory [11] is *maginally irrelevant.*

In the fermionic QED$_3$ (1) the above simple procedure does not work because one cannot simultaneously rescale the kinetic energy for all fermion species. We therefore keep the anisotropy confined to the matter part of (1) and proceed from there. The two-point vertex function of the noninteracting theory for, say, $(1, \bar{1})$ Dirac fermions is

$$\Gamma^{(2)\mathrm{free}}_{1\bar{1}} = \gamma_0 k_0 + v_F \gamma_1 k_1 + v_\Delta \gamma_2 k_2 \quad (2)$$

and the corresponding Green function equals

$$G^n_0(k) = \frac{\sqrt{g^n}_{\mu\nu}\gamma_\mu k_\nu}{k_\mu g_{\mu\nu} k_\nu} \equiv \frac{\gamma^n_\mu k_\mu}{k_\mu g_{\mu\nu} k_\nu}. \quad (3)$$

In what follows we assume that both v_F and v_Δ are dimensionless and that eventually one of them can be chosen to be unity by the appropriate choice for the "speed of light." The anisotropy parameter $\alpha_D = v_F/v_\Delta \neq 1$ breaks the Lorentz invariance of the theory (1). However, the theory still respects time reversal and parity and for $N > N_c$ the system is in its chirally symmetric phase. These symmetries force the fermion self-energy of the *interacting* theory to assume the following form:

$$\Sigma_{1\bar{1}} = A(k_{1\bar{1}}, k_{2\bar{2}})(\gamma_0 k_0 + v_F\zeta_1\gamma_1 k_1 + v_\Delta\zeta_2\gamma_2 k_2), \quad (4)$$

where $k_{n\bar{n}} = k_\mu g^n_{\mu\nu} k_\nu$. The coefficients ζ_i are in general different from unity. Furthermore, there is a discrete spatial symmetry which relates flavors $(1, \bar{1})$ and $(2, \bar{2})$ to the x and y directions in such a way that

$$\Sigma_{2\bar{2}} = A(k_{2\bar{2}}, k_{1\bar{1}})(\gamma_0 k_0 + v_\Delta\zeta_2\gamma_1 k_1 + v_F\zeta_1\gamma_2 k_2). \quad (5)$$

In the computation of the fermion self-energy, this discrete symmetry allows us to concentrate on a particular pair of nodes without any loss of generality.

Next, we turn to the gauge field propagator. We first work in the Lorentz gauge ($k_\mu a_\mu = 0$) and then extend our results to a general covariant gauge. To one-loop order the fermionic "screening" of the gauge field is given by the polarization function

$$\Pi_{\mu\nu}(k) = \frac{N}{2}\sum_{n=1,2}\int\frac{d^3q}{(2\pi)^3}\mathrm{Tr}[G^n_0(q)\gamma^n_\mu G^n_0(q+k)\gamma^n_\nu]. \quad (6)$$

The above expression can be evaluated by observing that it reduces to the isotropic $\Pi_{\mu\nu}(k)$ once the integrals are properly rescaled [4,5]. The result is

$$\Pi_{\mu\nu}(k) = \sum_n \frac{N}{16 v_F v_\Delta}\sqrt{k_\alpha g^n_{\alpha\beta} k_\beta}\left(g^n_{\mu\nu} - \frac{g^n_{\mu\rho}k_\rho g^n_{\nu\lambda}k_\lambda}{k_\alpha g^n_{\alpha\beta}k_\beta}\right), \quad (7)$$

where we have taken the advantage of the "nodal" metric $g^n_{\mu\nu}$ (1). This expression is explicitly transverse, i.e., $k_\mu\Pi_{\mu\nu}(k) = \Pi_{\mu\nu}(k)k_\nu = 0$, and symmetric in its space-time indices. It also properly reduces to the isotropic expression when $v_F = v_\Delta = 1$.

As opposed to the isotropic case, it is not quite as straightforward to determine the gauge field propagator $D_{\mu\nu}$. To proceed we first integrate out the fermions and expand the effective action to one-loop order

$$\mathcal{L}_{\mathrm{eff}}[a_\mu] = \tfrac{1}{2}(\Pi^{(0)}_{\mu\nu} + \Pi_{\mu\nu})a_\mu(k)a_\nu(-k), \quad (8)$$

where the bare gauge field stiffness is

$$\Pi^{(0)}_{\mu\nu} = e^2k^2\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right). \quad (9)$$

Now we introduce the dual field $b_\mu = \epsilon_{\mu\nu\lambda}q_\nu a_\lambda$, which is related to the physical fluctuating vorticity in the theory of Refs. [4,5]. We are free to integrate over b_μ with the restriction that it is transverse ($k_\mu b_\mu = 0$). Note that

$$\mathcal{L}_{\mathrm{eff}}[b_\mu] = \chi_0 b_0^2 + \chi_1 b_1^2 + \chi_2 b_2^2, \quad (10)$$

where $\{\chi_\mu\}$ are functions of k_μ:

$$\chi_\mu = \frac{1}{2e^2} + \frac{N}{32 v_F v_\Delta}\sum_{n=1,2}\frac{g^n_{\nu\nu}g^n_{\lambda\lambda}}{\sqrt{k_\alpha g^n_{\alpha\beta}k_\beta}}; \quad \mu \neq \nu \neq \lambda \in \{0,1,2\}. \quad (11)$$

At low energies we can neglect the nondivergent bare stiffness and thus we set $1/e^2 = 0$ in the above expression.

The expression (10) is manifestly gauge invariant and has the merit of not only being quadratic but also diagonal in the individual components of b_μ which greatly simplifies the computation of the b_μ correlation function:

$$2\langle b_\mu b_\nu\rangle = \frac{\delta_{\mu\nu}}{\chi_\mu} - \frac{k_\mu k_\nu}{\chi_\mu\chi_\nu}\left(\sum_i\frac{k_i^2}{\chi_i}\right)^{-1}. \quad (12)$$

The repeated indices are not summed over in the above expression.

After this little trick with the integration over b_μ, it is now quite simple to compute the propagator for the original gauge field a_μ and in the Lorentz gauge we obtain

$$D_{\mu\nu}(q) = \langle a_\mu a_\nu\rangle = \epsilon_{\mu\alpha\beta}\epsilon_{\nu\lambda\rho}\frac{q_\alpha q_\lambda}{q^4}\langle b_\beta b_\rho\rangle. \quad (13)$$

By employing the transverse character of $\langle b_\mu b_\nu\rangle$ (which is independent of the gauge) the above expression can be further reduced to

$$D_{\mu\nu}(q) = \frac{1}{q^2}\left[\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\langle b^2\rangle - \langle b_\mu b_\nu\rangle\right]. \quad (14)$$

We use the above result to define a general "covariant" gauge for the anisotropic theory as

$$D_{\mu\nu}(q) = \frac{1}{q^2}\left\{\left[\delta_{\mu\nu} - \left(1 - \frac{\xi}{2}\right)\frac{q_\mu q_\nu}{q^2}\right]\langle b^2\rangle - \langle b_\mu b_\nu\rangle\right\}, \quad (15)$$

where ξ is a continuous gauge fixing parameter. This expression is justified by the Fadeev-Popov procedure applied to the Lagrangian

$$\mathcal{L}_{\mathrm{eff}}[a_\mu] = \frac{1}{2}\left(\Pi_{\mu\nu} + \frac{1}{\xi}\frac{2k^2}{\langle b^2\rangle}\frac{k_\mu k_\nu}{k^2}\right)a_\mu(k)a_\nu(-k). \quad (16)$$

Note that $\langle b^2\rangle$ can be determined without ever considering the gauge fixing terms. The expression (15) is our final result for the gauge field propagator in an anisotropic covariant gauge.

Having determined the free fermion and screened gauge field propagators of the anisotropic theory, we can now compute the Dirac fermion self-energy generated by the photon exchange to leading order in $1/N$:

$$\Sigma_n(q) = \int\frac{d^3k}{(2\pi)^3}\gamma^n_\mu G^n_0(q-k)\gamma^n_\nu D_{\mu\nu}(k), \quad (17)$$

where n is the node index. After some tedious algebra this can be manipulated into

$$\Sigma_n(q) = -\sum_{\mu} \eta_\mu^n (\gamma_\mu^n q_\mu) \ln\left(\frac{\Lambda}{\sqrt{q_\alpha g_{\alpha\beta}^n q_\beta}}\right). \tag{18}$$

Here Λ is the ultraviolet cutoff and the coefficients η_μ are functions of the bare anisotropy which can be reduced to quadratures. In the case of weak anisotropy ($v_F = 1 + \delta,\ v_\Delta = 1$) to second order in δ:

$$\eta_0^{1\bar{1}} = -\frac{8}{3\pi^2 N}\left[1 - \frac{3}{2}\xi - \frac{1}{35}(40 - 7\xi)\delta^2\right), \tag{19}$$

$$\eta_1^{1\bar{1}} = -\frac{8}{3\pi^2 N}\left[1 - \frac{3}{2}\xi + \frac{6}{5}\delta - \frac{1}{35}(43 - 7\xi)\delta^2\right], \tag{20}$$

$$\eta_2^{1\bar{1}} = -\frac{8}{3\pi^2 N}\left[1 - \frac{3}{2}\xi - \frac{6}{5}\delta - \frac{1}{35}(1 - 7\xi)\delta^2\right]. \tag{21}$$

In the isotropic limit ($v_F = v_\Delta = 1$) we regain $\eta_\mu^n = -8(1 - \frac{3}{2}\xi)/3\pi^2 N$ as previously found by others [9,10].

We are now in position to turn to our main concern: the effect of anisotropy at the above critical point of isotropic QED_3. Before plunging into formal analysis, we first make some general physical observations regarding the RG flow of the anisotropy. First, by examining Eq. (18) it is clear that if $\eta_1^n = \eta_2^n$, then the anisotropy does not flow and remains equal to its bare value. This would imply that anisotropy is marginal and the theory is described by some anisotropic fixed point. For this to happen, however, there would have to be a symmetry that preserves the equality $\eta_1^n = \eta_2^n$. In the isotropic QED_3 the symmetry that protects the equality of η_μ's is the (Euclidean) Lorentz invariance. In our anisotropic case this symmetry is broken and thus we generically have $\eta_1^n \neq \eta_2^n$. Therefore, the anisotropy *runs* in the RG sense and *flows away* from its bare value. If we start with $\alpha_D > 1$ and find that $\eta_2^{1\bar{1}} > \eta_1^{1\bar{1}}$ at some scale $p < \Lambda$, the anisotropy is *marginally irrelevant* and decreases towards unity as we move toward infrared. On the other hand, if $\eta_2^{1\bar{1}} < \eta_1^{1\bar{1}}$, then the anisotropy becomes *marginally relevant*, continues increasing beyond its bare value, and the theory ultimately flows into some *new, anisotropic* critical point.

The above arguments concerning $\eta_2^n - \eta_1^n$ and RG flows are on solid ground physically only if they can be made in a gauge-independent way. This condition appears compromised by the fact that η_μ^n's are gauge-dependent quantities and explicitly include the gauge fixing parameter ξ. However, the difference $\eta_2^n - \eta_1^n$ *is* itself *gauge invariant.* This is seen directly from Eqs. (19)–(21), where the ξ dependence of all η's is exactly the same. This cancellation of ξ dependent terms in $\eta_2^n - \eta_1^n$ occurs not only for $\delta \ll 1$ but is the general feature of η_μ^n's to all orders in anisotropy and for any choice of covariant gauge fixing. Therefore, the RG analysis that follows is fully gauge invariant as it should be.

The renormalized two-point vertex function is related to the "bare" vertex via a fermion field rescaling factor Z_ψ as $\Gamma_R^{(2)} = Z_\psi \Gamma^{(2)}$. It is natural to demand that at some renormalization scale p, $\Gamma_R^{(2)}(p)$ have the form (at nodes 1 and $\bar{1}$) $\Gamma_R^{(2)}(p) = \gamma_0 p_0 + v_F^R \gamma_1 p_1 + v_\Delta^R \gamma_2 p_2$, where v_F^R and v_Δ^R are the renormalized velocities. The above equation corresponds to our renormalization condition through which we can eliminate the cutoff dependence and compute the RG flows.

To order $1/N$ we can write

$$\Gamma_R^{(2)}(p) = Z_\psi \gamma_\mu^n p_\mu \left(1 + \eta_\mu^n \ln\frac{\Lambda}{p}\right), \tag{22}$$

where we have used the self-energy (18). Multiplying both sides by γ_0 and taking the trace determines the field strength renormalization:

$$Z_\psi = \frac{1}{1 + \eta_0^n \ln\frac{\Lambda}{p}} \approx 1 - \eta_0^n \ln\frac{\Lambda}{p}. \tag{23}$$

We can now determine the renormalized Fermi and gap velocities:

$$\frac{v_F^R}{v_F} \approx \left(1 - \eta_0^{1\bar{1}} \ln\frac{\Lambda}{p}\right)\left(1 + \eta_1^{1\bar{1}} \ln\frac{\Lambda}{p}\right) \approx 1 - (\eta_0^{1\bar{1}} - \eta_1^{1\bar{1}}) \ln\frac{\Lambda}{p} \tag{24}$$

and

$$\frac{v_\Delta^R}{v_\Delta} \approx \left(1 - \eta_0^{1\bar{1}} \ln\frac{\Lambda}{p}\right)\left(1 + \eta_2^{1\bar{1}} \ln\frac{\Lambda}{p}\right) \approx 1 - (\eta_0^{1\bar{1}} - \eta_2^{1\bar{1}}) \ln\frac{\Lambda}{p}. \tag{25}$$

The corresponding renormalized Dirac anisotropy is therefore

$$\alpha_D^R \equiv \frac{v_F^R}{v_\Delta^R} \approx \alpha_D\left[1 - (\eta_2^{1\bar{1}} - \eta_1^{1\bar{1}}) \ln\frac{\Lambda}{p}\right]. \tag{26}$$

The RG beta function for the anisotropy is given by

$$\beta_{\alpha_D} = \frac{d\alpha_D^R}{d\ln p} = \alpha_D(\eta_2^{1\bar{1}} - \eta_1^{1\bar{1}}). \tag{27}$$

In the case of weak anisotropy ($v_F = 1 + \delta,\ v_\Delta = 1$) the above expression can be determined analytically as an expansion in δ. Using Eqs. (20) and (21) we obtain

$$\beta_{\alpha_D} = \frac{8}{3\pi^2 N}\left(\frac{6}{5}\delta(1 + \delta)(2 - \delta) + \mathcal{O}(\delta^3)\right). \tag{28}$$

Note that this expression is independent of the gauge parameter ξ. For $0 < \delta \ll 1$ the β function is positive which means that anisotropy decreases in the IR and thus the anisotropic QED_3 scales to an isotropic QED_3. For

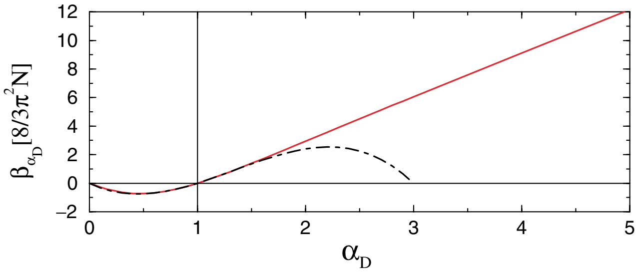

FIG. 1 (color online). The RG β-function for the Dirac anisotropy in units of $8/3\pi^2 N$. The solid line is the numerical integration, while the dash-dotted line is the analytical expansion around the small anisotropy [see Eqs. (19)–(21)]. At $\alpha_D = 1$, β_{α_D} crosses zero with positive slope, and therefore at large length scales the anisotropic QED_3 scales to an isotropic theory.

$-1 \ll \delta < 0$ the β function is negative and in this case α_D increases towards the fixed point $\alpha_D = 1$, i.e., again towards the isotropic QED_3. Note that for $\delta > 2$, $\beta < 0$ which may naively indicate that there is a fixed point at $\delta = 2$; this however cannot be trusted as it is outside of the range of validity of the power expansion of η_μ. The numerical evaluation of the quadrature for η_μ shows that, apart from the isotropic fixed point and the unstable fixed point at $\alpha_D = 0$, β_{α_D} does not vanish (see Fig. 1). This indicates that to the leading order in $1/N$ expansion, the theory flows into the isotropic fixed point and relativistic invariance is dynamically restored in the IR limit.

In summary, we have considered the effect of Dirac anisotropy in the Euclidean QED_3 and have found it to be irrelevant in the renormalization group sense to the leading order in $1/N$ expansion [12,13]. We expect our results to be useful in the continuing study of QED_3-like effective theories in condensed matter physics.

We thank V. Gusynin for sharing his valuable insights on QED_3 and Professor I. F. Herbut, Professor D. V. Khveshchenko, and Professor G. E. Volovik for useful comments. This work was supported in part by NSF Grant No. DMR00-94981 (O. V. and Z. T.) and by NSERC and the A. P. Sloan Foundation (M. F.).

[1] D. H. Kim, P. A. Lee, and X.-G. Wen, Phys. Rev. Lett. **79**, 2109 (1997).

[2] W. Rantner and X.-G. Wen, Phys. Rev. Lett. **86**, 3871 (2001); W. Rantner and X.-G. Wen, cond-mat/0010378.

[3] K. Farakos and N. E. Mavromatos, Phys. Rev. B **57**, 3017 (1998).

[4] M. Franz and Z. Tešanović, Phys. Rev. Lett. **87**, 257003 (2001).

[5] M. Franz, Z. Tešanović and O. Vafek, cond-mat/0203333.

[6] Z. Tešanović, O. Vafek, and M. Franz, Phys. Rev. B **65**, 180511 (2002).

[7] I. F. Herbut, Phys. Rev. Lett. **88**, 047006 (2002).

[8] M. Reenders, cond-mat/0110168.

[9] R. D. Pisarski, Phys. Rev. D **29**, 2423 (1984); T. W. Appelquist *et al.*, Phys. Rev. D **33**, 3704 (1986); M. R. Pennington and D. Walsh, Phys. Lett. B **253**, 246 (1991); I. J. R. Aitchison and N. E. Mavromatos, Phys. Rev. B **53**, 9321 (1996); P. Maris, Phys. Rev. D **54**, 4049 (1996).

[10] V. Gusynin, A. Hams, and M. Reenders, Phys. Rev. D **63**, 045025 (2001).

[11] I. F. Herbut and Z. Tešanović, Phys. Rev. Lett. **76**, 4588 (1996); Z. Tešanović, Phys. Rev. B **59**, 6449 (1999).

[12] Similar results have been obtained by D. Lee and I. F. Herbut (unpublished).

[13] This is reminiscent to what happens in higher dimensional gauge theories. For example, see S. Chadha and H. B. Nielsen, Nucl. Phys. **B217**, 125 (1983); J. Alexandre, K. Farakos, G. Koutsoumbas, and N. E. Mavromatos, Phys. Rev. D **64**, 125007 (2001); relativity as an emergent phenomenon in effective quantum field theories was considered by G. E. Volovik, JETP Lett. **73**, 162 (2001), and references therein.